The Distortion of Frequency-Modulated Waves by Transmission Networks*

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Summary—**A general solution to the problem of calculating the distortion imposed on the instantaneous frequency of a frequency-modulated wave in passing through a transmission network is obtained by a direct operational method. Approximate formulas for the cases of large and small deviation ratios are derived, and it is shown that a range of overlap exists in practical cases. For very large deviation ratios the distortion is entirely nonlinear in character and depends on the maximum frequency deviation, while for very small deviation ratios the distortion is entirely linear and is independent of the frequency deviation. The nature of the distortion is examined with particular reference to intermodulation distortion. When the modulating wave consists of two sine waves of different amplitudes and frequencies, intermodulation distortion takes the form of a frequency modulation of the small high-frequency component by the large low-frequency one. The application of negative feedback to a frequency-modulation receiver is considered. Numerical examples are worked out.**

I. Introduction

THE DISTORTION suffered by a frequency-modulated wave in passing through a transmission network has been investigated by Carson and Fry,[1] who obtained a theoretical solution, and more recently by Jaffe,[2] who calculated the numerical value of the harmonic distortion for the particular cases of sinusoidal modulation with networks consisting of either a single resonant circuit, or a pair of resonant circuits critically coupled and tuned to the carrier frequency.

These analyses apply to the case of a large deviation ratio, but important practical cases exist in which the deviation ratio is small; for example, a superheterodyne receiver in which negative feedback is used to reduce the frequency deviation of the received wave. Moreover, in practice, transmission networks are more complicated than those examined by Jaffe, and it is also desirable to know the distortion produced when modulating waves more complex than a single sine wave are used.

It is the object of the following analysis to derive formulas, suitable for both large and small deviation ratios, from which numerical values of the distortion products can be calculated for any type of transmission network.

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[1] J. R. Carson, and T. C. Fry, "Variable frequency electric circuit theory with application to the theory of frequency modulation," *Bell Sys. Tech. Jour.*, vol. 16, pp. 513–541; October, 1937.

[2] D. L. Jaffe, "A theoretical and experimental investigation of tuned circuit distortion in frequency-modulation systems," PROC. I.R.E., vol. 33, pp. 318–334; May, 1945.

II. List of Symbols

$A(u)$ = amplitude characteristic of the network (nepers)
D = deviation ratio = maximum frequency deviation/highest modulating frequency
$\phi(u)$ = phase characteristic of the network (radians)
p = differential operator = d/dt
$P(u)$ = in-phase component of the network transfer characteristic
$Q(u)$ = quadrature component of the network transfer characteristic
S = the modulating wave
$T(u) = Y(j\omega)$
$u = (\omega - \omega_c)/\omega_B$
v_i = input to a network
v_0 = output from a network
ω_B = semibandwidth of the network (radians/second)
ω_c = carrier frequency (radians/second)
$\Delta\omega$ = maximum frequency deviation (radians/second)
$Y(j\omega)$ = steady-state complex transfer characteristic of the network = output/input.

III. The General Solution

Let the modulating wave be denoted by S, and let the peak value of S be unity. A sinusoidal carrier wave frequency modulated by S can be written

$$v_i = \cos\left(\omega_c t + \Delta\omega \int S dt\right)$$

$$= R \exp j\left(\omega_c t + \Delta\omega \int S dt\right)$$

where ω_c is the carrier frequency, $\Delta\omega$ is the maximum frequency deviation, and R denotes "the real part of." Conventionally, R is omitted from the analysis, but it should always be understood.

When the modulated carrier is applied at the input terminals of a linear transmission network having a transfer characteristic $Y(j\omega)$, the output from the network is

$$v_0 = Y(p)v_i.$$

$Y(p)$ is the transfer characteristic with the differential operator $p = (d/dt)$ written in place of $j\omega$. $Y(p)$ is an operational function, and the output v_0 is obtained by operating on v_i with $Y(p)$. This can be done conveniently by using Murphy's shifting theorem[3] to give the result

$$v_0 = \exp j\left(\omega_c t + \Delta\omega \int S dt\right) Y(p + j\omega_c + j\Delta\omega S). \quad (1)$$

The subject of the operational function is now unity. It is supposed that the modulated carrier has been applied to the network for a very long time, so that the transient solution $Y(p)\cdot 0$ has vanished at the time under consideration.

Since most of the networks to be analyzed are band-pass filters, it is convenient to express a frequency in terms of its difference $\omega - \omega_c$ from the carrier frequency, and also to express this difference as a fraction u of the filter semibandwith ω_B, i.e., $u = (\omega - \omega_c)/\omega_B$. The transfer characteristic may then be specified in terms of a shape function $T(u)$, a scale factor ω_B, and a position factor ω_c, thus:

$$Y(j\omega) = Y(j\omega_c + j\,\overline{\omega - \omega_c}) = Y(j\omega_c + ju\omega_B) = T(u). \quad (2)$$

Applying this transformation to (1) gives

$$Y(p + j\omega_c + j\Delta\omega S) = T\left(\frac{\Delta\omega S - jp}{\omega_B}\right).$$

If it is assumed that this function can be expanded in a power series by Maclaurin's theorem, then

$$T\left(\frac{\Delta\omega S - jp}{\omega_B}\right) = \sum_0^\infty \frac{1}{n!}\left(\frac{\Delta\omega S - jp}{\omega_B}\right)^n T^n(0) \quad (3)$$

where $T^n(0) = d^n/dt^n T(u)\big|_u = 0$. The expansion is valid only if the series so obtained converges. In particular, the series does not converge if S is a step function, or if there are any discontinuities in $T(u)$ or its derivatives.

The operator $(\Delta\omega S - jp)^n$ denotes $(\Delta\omega S - jp)(\Delta\omega S - jp) \cdots$ to n terms, and may be expanded as follows:

$$(\Delta\omega S - jp) = \Delta\omega S$$

$$(\Delta\omega S - jp)^2 = (\Delta\omega S - jp)\Delta\omega S = (\Delta\omega S)^2 - j\Delta\omega S'$$

and so on:

$$\left(S' = \frac{d}{dt} S\right).$$

It is found that the terms in the series resulting from the expansion of (3) are themselves the Maclaurin series for functions such as $T(\Delta\omega S/\omega_B)$. When all the terms are collected in this way, (1) can be written

[3] A. G. Warren, "Mathematics Applied to Electrical Engineering," Chapman and Hall, London, 1939; p. 169. The theorem states that $Y(p) \exp f(t) \equiv \exp f(t)\, Y(p + f'(t))$ both sides of the identity being operational functions. The analysis can also be carried out, with a little more trouble, by using the well-known Heaviside shifting theorem, which is a particular case of Murphy's theorem. This was the method used by Jaffe.

$$\begin{aligned} v_0 = \Bigg[\exp j\left(\omega_c t + \Delta\omega \int S dt\right)\Bigg]\Bigg[T\left(\frac{\Delta\omega S}{\omega_B}\right) \\ - \frac{\Delta\omega}{\omega_B}\left\{\frac{jS'}{2!\omega_B} T''\left(\frac{\Delta\omega S}{\omega_B}\right) + \frac{S''}{3!\omega_B^2} T'''\left(\frac{\Delta\omega S}{\omega_B}\right)\right. \\ \left. - \frac{jS'''}{4!\omega_B^3} T^{IV}\left(\frac{\Delta\omega S}{\omega_B}\right) - \frac{S^{IV}}{5!\omega_B^4} T^{V}\left(\frac{\Delta\omega S}{\omega_B}\right) + \cdots\right\} \\ + j\frac{\Delta\omega^2}{2\omega_B^3} S'\left\{\frac{jS'}{4\omega_B} T^{IV}\left(\frac{\Delta\omega S}{\omega_B}\right)\right. \\ + \frac{S''}{3!\omega_B^2} T^{V}\left(\frac{\Delta\omega S}{\omega_B}\right) - \frac{jS'''}{4!\omega_B^3} T^{VI}\left(\frac{\Delta\omega S}{\omega_B}\right) \\ \left. - \frac{S^{IV}}{5!\omega_B^4} T^{VII}\left(\frac{\Delta\omega S}{\omega_B}\right) + \cdots\right\} \quad (4) \\ + \text{etc.}\Bigg] \end{aligned}$$

This is the general solution, but in the form given above it is of little practical use. The desirable form of solution would express the result as a carrier wave modulated in amplitude and frequency. Approximate solutions of this form can be found when the deviation ratio is large or small.

IV. Solution for Large Deviation Ratios

When the deviation ratio is large, S'/ω_B is small, and the terms of the series in (4) are of rapidly decreasing magnitude. Only the first two terms need therefore be considered, and, if D is very large, only the first term.

It is convenient to express the transfer characteristic in polar form: $T(u) = \exp\{A(u) + j\phi(u)\}$, where $A(u)$ is the amplitude characteristic (nepers), and $\phi(u)$ is the phase characteristic (radians). The second derivative $T''(u)$ is easily found and a common factor $T(\Delta\omega S/\omega_B)$ can be removed from the first two terms of (4), which can then be written (neglecting terms beyond the second) as

$$\begin{aligned} v_0 = \Bigg[\exp A\left(\frac{\Delta\omega S}{\omega_B}\right) \exp j\left\{\omega_c t + \Delta\omega \int S dt \right. \\ \left. + \phi\left(\frac{\Delta\omega S}{\omega_B}\right)\right\}\Bigg]\Bigg[1 + \frac{\Delta\omega S'}{2\omega_B^2}\left\{\phi''\left(\frac{\Delta\omega S}{\omega_B}\right)\right. \\ + 2A'\left(\frac{\Delta\omega S}{\omega_B}\right)\phi'\left(\frac{\Delta\omega S}{\omega_B}\right) - jA''\left(\frac{\Delta\omega S}{\omega_B}\right) \\ \left. - j\left[A'\left(\frac{\Delta\omega S}{\omega_B}\right)\right]^2 + j\left[\phi'\left(\frac{\Delta\omega S}{\omega_B}\right)\right]^2\right\}\Bigg]. \end{aligned}$$

Of the series of terms in the square brackets, the imaginary part is small compared with 1, and the variable part of the real part is also small compared with 1. The series may therefore be replaced by $K \exp j\alpha$, where K is the real part, and α the imaginary part of the series. Then

$$v_0 = M \exp j\left(\omega_c t + \Delta\omega \int \omega_f dt\right)$$

where

$$M = \left[\exp A\left(\frac{\Delta\omega S}{\omega_B}\right)\right]\left[1 + \frac{\Delta\omega S'}{2\omega_B{}^2}\left\{\phi''\left(\frac{\Delta\omega S}{\omega_B}\right) + 2A'\left(\frac{\Delta\omega S}{\omega_B}\right)\phi'\left(\frac{\Delta\omega S}{\omega_B}\right)\right\}\right]$$

$$\omega_d = S + \frac{1}{\Delta\omega}\frac{d}{dt}\cdot\phi\left(\frac{\Delta\omega S}{\omega_B}\right) - \frac{1}{2\omega_B{}^2}\frac{d}{dt}\left[S'\left\{A''\left(\frac{\Delta\omega S}{\omega_B}\right) + \left[A'\left(\frac{\Delta\omega S}{\omega_B}\right)\right]^2 - \left[\phi'\left(\frac{\Delta\omega S}{\omega_B}\right)\right]^2\right\}\right]. \tag{5}$$

M is the amplitude and $\Delta\omega\omega_j$ the frequency deviation of v_0.

If the transfer characteristics can be represented by simple functions and the modulating wave is also simple, it is sometimes possible to calculate the distortion directly from (5). In general, however, this is not possible, and the transfer functions have to be expressed in a form amenable to computation, e.g., a power series expansion, thus:

$$A(u) = A_0 + uA_1 + \frac{u^2}{2!}A_2 + \cdots$$
$$\phi(u) = \phi_0 + u\phi_1 + \frac{u^2}{2!}\phi_2 + \cdots. \tag{6}$$

The quantity ϕ_1 is the coefficient of the linear part of the phase characteristic, and thus represents time delay for the whole wave. It is advantageous to proceed as if ϕ_1 were zero and to correct the final result, if required, for the time delay corresponding to ϕ_1. This reduces the number of terms to be handled. The coefficients A_0 and ϕ_0, which represent a constant amplitude change and a constant phase shift of the carrier, may also without error be equated to zero.

The series expansion should represent the characteristic accurately over a sufficient range of u; namely, a range corresponding to frequencies slightly beyond the frequency excursion of the modulated carrier wave.

When the series given by (6) are substituted in (5), the distortion terms may be divided into three groups. First, a linear term which is simply a derivative of S. This is

$$-\frac{S''}{2\omega_B{}^2}(A_1{}^2 + A_2).$$

Next, even-order nonlinear terms,

$$\frac{1}{\Delta\omega}\frac{d}{dt}\left\{\frac{\phi_2}{2!}\left(\frac{\Delta\omega S}{\omega_B}\right)^2 + \frac{\phi_4}{4!}\left(\frac{\Delta\omega S}{\omega_B}\right)^4 + \cdots\right\}$$
$$-\frac{1}{2\omega_B{}^2}\frac{d}{dt}\left[S'\left\{\frac{\Delta\omega S}{\omega_B}(A_3 + 2A_1A_2) + \left(\frac{\Delta\omega S}{\omega_B}\right)^3\left(\frac{1}{6}A_5 + \frac{1}{3}A_4A_1 + A_3A_2 - \phi_3\phi_2\right) + \cdots\right\}\right]. \tag{7}$$

Finally, the odd-order nonlinear terms,

$$\frac{1}{\Delta\omega}\frac{d}{dt}\left\{\frac{\phi_3}{3!}\left(\frac{\Delta\omega S}{\omega_B}\right)^3 + \frac{\phi_5}{5!}\left(\frac{\Delta\omega S}{\omega_B}\right)^5 + \cdots\right\}$$
$$-\frac{1}{2\omega^2{}_B}\frac{d}{dt}\left[S'\left\{\left(\frac{\Delta\omega S}{\omega_B}\right)^2\left(\frac{1}{2}A_4 + A_3A_1 + A_2{}^2 - \phi_2{}^2\right) + \left(\frac{\Delta\omega S}{\omega_B}\right)^4\left(\frac{1}{24}A_6 + \frac{1}{12}A_5A_1 + \frac{1}{3}A_4A_2 + \frac{1}{4}A_3{}^2 - \frac{1}{3}\phi_4\phi_2 - \frac{1}{4}\phi_3{}^2\right)\right\}\right]. \tag{8}$$

If the amplitude characteristic is symmetrical and the phase characteristic skew-symmetrical, the even-order terms and some coefficients of the odd-order terms vanish.

V. Solution for Small Deviation Ratios

To obtain the formula for small deviation ratios it is convenient to express the transfer characteristic in Cartesian form: $T(u) = P(u) + jQ(u)$. $P(u)$, and $Q(u)$ are the in-phase and quadrature components, respectively. Equation (4) may then be written

$$v_0 = \left[\exp j\left(\omega_c t + \Delta\omega\int S dt\right)\right][1 + R + jI]$$

$$R = P\left(\frac{\Delta\omega S}{\omega_B}\right) - 1 + \frac{\Delta\omega}{\omega_B}\left\{\frac{S'}{2!\omega_B}Q''\left(\frac{\Delta\omega S}{\omega_B}\right) - \frac{S''}{3!\omega_B{}^2}P'''\left(\frac{\Delta\omega S}{\omega_B}\right) - \cdots\right\}$$
$$- \frac{\Delta\omega^2 S'}{2\omega_B{}^3}\left\{\frac{S'}{4\omega_B}P^{IV}\left(\frac{\Delta\omega S}{\omega_B}\right) + \frac{S''}{3!\omega_B{}^2}Q^{V}\left(\frac{\Delta\omega S}{\omega_B}\right) - \cdots\right\}$$

$$I = Q\left(\frac{\Delta\omega S}{\omega_B}\right) - \frac{\Delta\omega}{\omega_B}\left\{\frac{S'}{2!\omega_B}P''\left(\frac{\Delta\omega S}{\omega_B}\right) + \frac{S''}{3!\omega_B}Q'''\left(\frac{\Delta\omega S}{\omega_B}\right) - \cdots\right\}$$
$$- \frac{\Delta\omega^2}{2\omega_B{}^3}S'\left\{\frac{S'}{4\omega_B}Q^{IV}\left(\frac{\Delta\omega S}{\omega_B}\right) - \frac{S''}{3!\omega_B{}^2}P^{V}\left(\frac{\Delta\omega S}{\omega_B}\right) - \cdots\right\}. \tag{9}$$

Since the deviation ratio is small, $\Delta\omega/\omega_B$ is small. Also, from the relations between the polar and Cartesian forms of the transfer characteristic given in the

Appendix, $P(0)=1$ when $A(0)=0$, the condition assumed in the previous section. It follows that both R and I are small, compared with 1.

$$\therefore \quad 1+R+jI$$
$$=\{(1+R)^2+I^2\}^{1/2}\exp j\{\tan^{-1} I/(1+R)\}$$
$$\doteq (1+R)\exp jI(1-R).$$

Equation (4) then becomes

$$v_0=(1+R)\exp j\left(\omega_c t+\Delta\omega\int \omega_d dt\right)$$

where

$$\omega_d=S+\frac{1}{\Delta\omega}\frac{d}{dt}\cdot I(1-R). \qquad (10)$$

It is now assumed that $P(u)$ and $Q(u)$ can be expressed in the form of power series

$$P(u)=P_0+uP_1+\frac{u^2}{2!}P_2+\cdots$$

$$Q(u)=Q_0+uQ_1+\frac{u^2}{2!}Q_2+\cdots. \qquad (11)$$

In the previous section it was shown that it is permissible and advantageous to write $A_0=\phi_0=\phi_1=0$. From the relations given in the Appendix, the corresponding conditions for the Cartesian form are $P_0=1$ $Q_0=Q_1=0$.

On substituting the series of (11) into the expressions for R and I given by (9), the frequency deviation ω_d can be expanded in a series. Since $\Delta\omega/\omega_B$ is small, only terms with coefficients proportional to $\Delta\omega/\omega_B$ and $(\Delta\omega/\omega_B)^2$ need be considered in addition to terms independent of $\Delta\omega$. The linear distortion terms are

$$-\frac{S''P_2}{2!\omega_B^2}-\frac{S'''Q_3}{3!\omega_B^3}+\frac{S^{IV}P_4}{4!\omega_B^4}+\cdots.$$

The even-order nonlinear terms are

$$\frac{\Delta\omega}{\omega_B^2}\frac{d}{dt}\left[S\left\{\frac{SQ_2}{2}-\frac{S'}{2!\omega_B}(P_3-P_2P_1)-\frac{S''}{3!\omega_B^2}(Q_4-Q_3P_1)\right.\right.$$
$$\left.\left.+\frac{S'''}{4!\omega_B^3}(P_5-P_4P_1)+\cdots\right\}\right]$$
$$-\frac{\Delta\omega}{2\omega_B^3}\frac{d}{dt}\left[S'\left\{\frac{S'}{2!\omega_B}\left(\frac{1}{2}Q_4-P_2Q_2\right)\right.\right.$$
$$-\frac{S''}{3!\omega_B^2}(P_5+Q_3Q_2-P_3P_2)$$
$$\left.\left.-\frac{S'''}{4!\omega_B^3}(Q_6-P_4Q_2-Q_4P_2)+\cdots\right\}\right] \qquad (12)$$

and the odd-order nonlinear terms are

$$\frac{1}{2}\frac{\Delta\omega^2}{\omega_B^3}\frac{d}{dt}\left[S^2\left\{S\left(\frac{1}{3}Q_3-Q_2P_1\right)\right.\right.$$
$$-\frac{S'}{2!\omega_B}(P_4-P_2^2-2P_3P_1+Q_2^2)$$
$$-\frac{S''}{3!\omega_B^2}(Q_5-Q_3P_2-2Q_4P_1-P_3Q_2)$$
$$\left.\left.+\frac{S'''}{4!\omega_B^3}(P_6-P_4P_2-2P_5P_1+Q_4Q_2)+\cdots\right\}\right]$$
$$-\frac{1}{2}\frac{\Delta\omega^2}{\omega_B^4}\frac{d}{dt}\left[SS'\left\{\frac{S'}{2!\omega_B}\left(\frac{1}{2}Q_5-Q_3P_2\right.\right.\right.$$
$$\left.-\frac{1}{2}Q_4P_1-P_3Q_2\right)$$
$$-\frac{S''}{3!\omega_B^2}(P_6-P_4P_2+Q_3^2-P_5P_1+Q_4Q_2-P_3^2)$$
$$-\frac{S'''}{4!\omega_B^3}(Q_7-Q_5P_2-P_4Q_3$$
$$\left.\left.-Q_6P_1-P_5Q_2-Q_4P_3)+\cdots\right\}\right]. \qquad (13)$$

The first term in each of the series in (13) has an anomalous value, but all the following terms form a regular sequence.

When the in-phase characteristic is symmetrical and the quadrature characteristic skew-symmetrical, the even-order terms and half of the coefficients of the odd-order terms vanish.

VI. Discussion of Results

For both large and small deviation ratios, the amplitudes of the linear distortion terms are independent of the frequency deviation. These terms which represent phase and frequency distortion of the modulating wave are not usually of interest.

For large deviation ratios, the distortion given by (5) can be divided into two parts. The first and principal part, $(1/\Delta\omega)\cdot(d/dt)\phi(\Delta\omega S/\omega_B)$, depends only on the phase characteristic; the second part is determined mainly by the amplitude characteristic and partly by the phase characteristic. As the deviation ratio is large, S'/ω_B is small, so that the distortion produced by the amplitude characteristic is usually small compared with that produced by the phase characteristic. Moreover, if the modulating wave is the sum of a number of cosine waves, it is clear from (5) that the principal part of the distortion is the sum of a number of sine waves, whereas the second part is the sum of a number of cosine waves. Thus, the distortion products due to the amplitude characteristic are in phase quadrature with the principal distortion products produced by the phase characteristic, and so have little effect on the total distortion until their magnitude equals or exceeds that of the principal part of the distortion.

If the deviation ratio D is increased while $\Delta\omega/\omega_B$ is kept constant, the principal part of the distortion decreases in the ratio $1/D$ and the amplitude characteristic distortion in the ratio $1/D^2$. As the deviation ratio is increased, therefore, the distortion is ultimately entirely nonlinear in character, and is produced by the nonlinear part of the phase characteristic.

For small deviation ratios the limits of the frequency spectrum of a modulated carrier wave are determined by the spectrum of the modulating wave rather than by the maximum frequency deviation. Consequently, ω_B cannot be reduced below a certain minimum value. From the formulas of Section V it is then seen that as $\Delta\omega$ is reduced the distortion is ultimately entirely linear in character, is independent of the maximum frequency deviation, and is produced by the symmetrical part of the in-phase characteristic and the skew-symmetrical part of the quadrature characteristic.

VII. Nature of the Distortion in a Frequency-Modulation Network

If the input to a nonlinear vacuum-tube amplifier with a resistive load is the sum for a number of cosine waves, the output from the amplifier, including the distortion products, is also the sum of a number of cosine waves. In the previous section it was noted that the principal distortion products for large deviation ratios are the sum of a number of sine waves. Thus, in a frequency-modulation network the principal distortion products are in phase quadrature with the corresponding distortion products in a nonlinear amplifier.

A convenient method of specifying and measuring distortion in low-frequency apparatus is the intermodulation method, in which two sine waves, one of low frequency and amplitude nearly equal to the capacitance of the apparatus, the other of high frequency and small amplitude, are applied simultaneously to the apparatus. Nonlinear distortion results in the amplitude of the high-frequency component being modulated by the low-frequency component, the amount of such modulation being a measure of the distortion. In frequency-modulation systems, however, the intermodulation distortion is of an entirely different character.

Let $S = C_1 \cos \omega_1 t + C_2 \cos \omega_2 t$ where $C_2 \ll C_1$ and $\omega_1 \ll \omega_2$, and suppose that the deviation ratio is large so that only the principal part of the phase-characteristic distortion is significant. Then, from (5), the frequency deviation is

$$C_1 \cos \omega_1 t + C_2 \cos \omega_2 t + \frac{1}{\Delta\omega}\frac{d}{dt}\phi\left(\frac{\Delta\omega}{\omega_B} C_1 \cos \omega_1 t + \frac{\Delta\omega}{\omega_B} C_2 \cos \omega_2 t\right).$$

Since C_2 is small, the last term can be written approximately (because $C_1 + C_2 = 1$) as

$$\frac{1}{\Delta\omega}\frac{d}{dt}\phi\left(\frac{\Delta\omega}{\omega_B}\cos \omega_1 t\right) + \frac{C_2}{\omega_B}\frac{d}{dt}\left\{\cos \omega_2 t\, \phi'\left(\frac{\Delta\omega}{\omega_B}\cos \omega_1 t\right)\right\}.$$

The first term in this expression represents harmonics of the low-frequency component. The second term represents the intermodulation products, and, since $\omega_2 \gg \omega_1$, this term is

$$-\frac{C_2 \omega_2}{\omega_B}\sin \omega_2 t\, \phi'\left(\frac{\Delta\omega}{\omega_B}\cos \omega_1 t\right).$$

Adding to this the component of frequency ω_2 in the frequency deviation gives

$$C_2\left[\cos \omega_2 t - \frac{\omega_2}{\omega_B}\sin \omega_2 t\, \phi'\left(\frac{\Delta\omega}{\omega_B}\cos \omega_1 t\right)\right]$$
$$\doteq C_2 \cos\left\{\omega_2 t + \frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos \omega_1 t\right)\right\}. \qquad (14)$$

Nonlinear distortion is manifest as a modulation of the high-frequency component, not in amplitude, but in frequency (or phase) by the low-frequency component. The amount of the modulation depends not only on the amplitude of the low-frequency component, but is also directly proportional to the frequency of the high-frequency component.

Since the intermodulation is of frequency instead of amplitude, the intermodulation products are in phase-quadrature with the corresponding components in the vacuum-tube-amplifier case. Listening tests have shown, as would be expected, that the ear is unable to distinguish this phase difference, provided the distortion is not too great. Accordingly, a sound wave given by (14) produces the same aural effect as a wave given by

$$C_2\left\{1 + \frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos \omega_1 t\right)\right\}\cos \omega_2 t$$

in which the high-frequency component $C_2 \cos \omega_2 t$ is modulated in amplitude.

It is, therefore, permissible to specify the distortion in a frequency-modulation network as intermodulation distortion, the magnitude being the maximum deviation of

$$1 + \frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos \omega_1 t\right)$$

from its mean value, but it must be remembered that the modulation is of frequency and cannot be measured by the same methods as are used for amplitude intermodulation. A suitable method is described in the Appendix.

VIII. Methods of Calculation and Examples

If the transfer characteristic can be represented by simple functions and the modulating wave is also simple, it is sometimes possible to calculate the distortion directly from expression (5). An example of such a calculation is given below. In general, however, this is not possible, and the transfer functions have to be expressed in power series form. The distortion is then calculated from (7), (8), (12), and (13).

If the power series for either the polar or Cartesian form of the transfer characteristic are given, the series for the other form may be obtained from the relations between coefficients given in the Appendix.

Finally, quantities of the form S^n have to be evaluated. For the purpose of analysis, a modulating wave which yields a fair amount of information without too much labor is the sum of two cosine (or sine) waves of different or equal amplitudes. A method of expanding the expression $(k_1 \cos \omega_1 t + k_2 \cos \omega_2 t)^n$ in a series of terms of the type $A_{pq} \cos (p\omega_1 \pm q\omega_2)t$ is given in the Appendix.

The intermodulation distortion is found by calculating the maximum or minimum value and the mean value of

$$\frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right).$$

Example 1

A high-frequency carrier wave, frequency-modulated by a 5-kc. cosine wave, is applied to a network consisting of a single parallel-resonant circuit such that the amplitude response is -3 db at frequencies differing by ± 25 kc. from the carrier frequency. Find the third-harmonic distortion in the frequency deviation of the output wave as the maximum deviation is varied from 10 to 100 kc.

The transfer characteristic of the network is $T(u) = (1+ju)^{-1} \exp ju$, and $\omega_B = 25$ kc. The factor $\exp ju$ is added to satisfy the condition that the phase characteristic should have no linear part. Then $A(u) = -\frac{1}{2}\log h(1+u^2)$ and $\phi(u) = u - \tan^{-1}u$.

Let the modulating wave be $\cos \omega_m t$. For large deviation ratios, equation (5) is used. Now $A''(u) + \{A'(u)\}^2 - \{\phi'(u)\}^2 = -(u^2-1)^2(u^2+1)^{-2}$, so that (5) becomes

$$\cos\omega_m t + \frac{1}{\Delta\omega}\cdot\frac{d}{dt}\left\{\frac{\Delta\omega}{\omega_B}\cos\omega_m t - \tan^{-1}\left(\frac{\Delta\omega}{\omega_B}\cos\omega_m t\right)\right\}$$

$$-\frac{\omega_m}{2\omega_B^2}\frac{d}{dt}\left[\sin\omega_m t\left\{\left(\frac{\Delta\omega^2}{\omega_B^2}\cos^2\omega_m t - 1\right)^2\right.\right.$$

$$\left.\left.\cdot\left(\frac{\Delta\omega^2}{\omega_B^2}\cos^2\omega_m t + 1\right)^{-2}\right\}\right]. \qquad (15)$$

Now $\tan^{-1}\left(\frac{\Delta\omega}{\omega_B}\cos\omega_m t\right)$ may be expanded in a Fourier series from the formula[4]

$$\tan^{-1}\left(\frac{2a\cos x}{1-a^2}\right) = 2\sum_1^\infty (-1)^{n-1}\frac{a^{2n-1}}{2n-1}\cos(2n-1)x$$

by writing

$$a = \left\{1 + \frac{\omega_B^2}{\Delta\omega^2}\right\}^{1/2} - \frac{\omega_B}{\Delta\omega}.$$

In the third term the factor

$$\left(\frac{\Delta\omega^2}{\omega_B^2}\cos^2\omega_m t + 1\right)^{-2}$$

may be expanded by the formula[5]

$$(1-b^2)^3(1+b^2+2b\cos x)^{-2}$$

$$= 1 + b^2 + 2\sum_1^\infty(-b)^n\{n+1-b^2(n-1)\}\cos nx$$

by writing $b = a^2$. It is then a matter of straightforward trigonometry to find the terms of third-harmonic frequencies in (15). These are

$$-\frac{2\omega_m}{\Delta\omega}b^{3/2}\sin 3\omega_m t$$

$$+\frac{3}{2}\left(\frac{\omega_m}{\omega_B}\right)^2\left\{1+\frac{1}{2}\frac{\Delta\omega^2}{\omega_B^2}\right\}^{-2}(1+b^2)^2(1-b^2)^{-3}$$

$$\cdot\left\{b(2-b)(1+b)^2 + \frac{1}{2}\frac{\Delta\omega^2}{\omega_B^2}(1-2b)(1-b^2)^2\right.$$

$$\left.-\frac{3}{16}\frac{\Delta\omega^4}{\omega_B^4}(1-b)^2(1-b^2)^2\right\}\cos 3\omega_m t. \qquad (16)$$

The amplitude of the resultant is the square root of the sum of the squares of the amplitudes of sine and cosine terms.

For small deviation ratios the Cartesian form of the transfer characteristic is used. If $(1+ju)^{-1}\exp ju$ is expanded in a series of powers of u, then

$$P(u) = 1 - \frac{u^2}{2!} + \frac{9u^4}{4!} - \frac{265u^6}{6!} + \frac{14833}{8!}u^8\cdots$$

$$Q(u) = \frac{2u^3}{3!} - \frac{44u^5}{5!} + \frac{1854}{7!}u^7\cdots.$$

Hence,

$$P_2 = -1 \quad P_4 = 9 \quad P_6 = -265 \quad P_8 = 14833$$

$$Q_3 = 2 \quad Q_5 = -44 \quad Q_7 = 1854.$$

On substituting these values into (13), the terms of third-harmonic frequency are found to be

[4] "Smithsonian Mathematical Formulae and Tables of Elliptic Functions," Smithsonian Institution, 1939; p. 140.
[5] J. Edwards, "The Integral Calculus," Macmillan Co., London, 1922; vol. 2, p. 303.

$$-\frac{\Delta\omega^2\omega_m}{4\omega_B^3}\left[1-\frac{51}{2}\frac{\omega_m^2}{\omega_B^2}+\frac{1077}{8}\frac{\omega_m^4}{\omega_B^4}\right]\sin 3\omega_m t$$

$$+\frac{3}{2}\frac{\Delta\omega^2\omega_m^2}{\omega_B^4}\left[1-\frac{79}{6}\frac{\omega_m^2}{\omega_B^2}+\frac{1409}{40}\frac{\omega_m^4}{\omega_B^4}\right]\cos 3\omega_m t. \quad (17)$$

Finally, putting $\omega_m=2\pi 5000$, $\omega_B=2\pi 25{,}000$, the amplitude of the third-harmonic component is $0.033(\Delta\omega^2/\omega_B^2)$.

On Fig. 1 is shown the amplitude of the third harmonic calculated from (16) and (17). The same figure shows the experimental and theoretical results obtained by Jaffe. Jaffe's theoretical values correspond to the sine term in (16), i.e., to the distortion due to the phase characteristic.

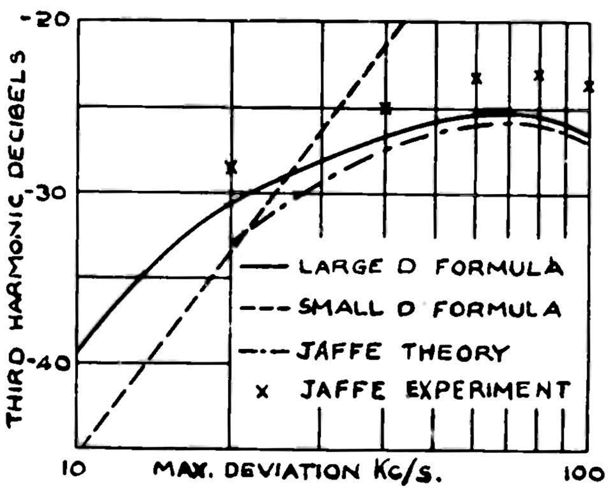

Fig. 1—Third-harmonic distortion in a single resonant circuit.

Example 2

The frequency-selective circuits of an amplifier consist of three identical band-pass filters connected in cascade via amplifier tubes. Each filter is made up of two identical simple resonant circuits critically coupled. The over-all amplitude response of the amplifier is −6 db at frequencies differing by ±100 kc. from the midband frequency.

Estimate the intermodulation distortion at a frequency of 12 kc. when $\Delta\omega=75$ kc.,

(a) when the carrier frequency is equal to the midband frequency; and

(b) when the carrier frequency differs by 25 kc. from the midband frequency.

A carrier wave modulated by two cosine waves of equal amplitudes and of frequencies 3 kc. and 5 kc. is applied to the amplifier. The maximum deviation is 75 kc.

(c) Find the amplitude of the distortion product of frequency 11 kc. when the carrier frequency is equal to the midband frequency.

(d) Find the amplitude of the distortion product of frequency 8 kc. when the carrier frequency differs by 25 kc. from the midband frequency.

The transfer characteristic for a pair of identical resonant circuits critically coupled is $(1-\frac{1}{2}\alpha^2u^2+j\alpha u)^{-1}\exp j\alpha u$, where α is a constant depending on the bandwidth. Taking ω_B as 100 kc., the transfer characteristic for the complete amplifier is $T(u)=(1-0.769u^2+j1.24u)^{-3}\exp j3.72u$, from which $\phi(u)=+3.72u-3\tan^{-1}\{1.24u/(1-0.769u^2)\}$. The Gregory series for $\tan^{-1}x$ converges for $|x|\leq 1$, corresponding to $|u|\leq 0.59$. By expressing $\phi(u)$ as an inverse sine, a series can be found which converges for $|u|\leq 1.14$, but the convergence is very slow. However, it is found that for values of u up to 1, $\phi(u)$ can be approximated by the expression $\phi(u)=-0.95u^3+0.51u^5-0.017u^7$.

When the carrier frequency is shifted from the midband frequency of the amplifier by 25 kc., this is equivalent to shifting the working point of the transfer characteristic from 0 to $25/100=\frac{1}{4}$. Consequently, the new phase characteristic is obtained by writing $u+\frac{1}{4}$ in place of u in the above expression for $\phi(u)$. Omitting the linear and constant terms, this is

$$-0.63u^2-0.63u^3+0.63u^4+0.49u^5-0.030u^6-0.017u^7.$$

The amplitude characteristic has a negligible effect on the distortion, and is therefore ignored.

The intermodulation distortion is the maximum deviation of

$$\frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right)=0.12\phi'(0.75\cos\omega_1 t),$$

from its mean value. In case (a) the mean value is, from (20), −0.061. The maximum value is obviously 0, and the minimum value is easily shown to be −0.096; the intermodulation distortion is, therefore, 6 per cent. In case (b) the mean value is −0.03, and the maximum and minimum values are 0.02 and −0.076. The intermodulation distortion is now 5 per cent.

Cases (c) and (d) are solved from (8) and (7), using the expressions for $\phi(u)$ given above, and evaluating the amplitudes of the distortion products from (20), with $p=2$, $q=1$ in case (c), and $p=q=1$ in case (d). The results are: 0.0038 and 0.0056. In case (c) the distortion products are all of the odd-order type, i.e., of the form $a_{pq}\cos(p\omega_1\pm q\omega_2)t$ where $p+q$ is odd, but in case (d) the even-order type is predominant.

IX. Distortion in a Receiver with Negative Feedback

Negative feedback may be applied to a superheterodyne receiver by arranging that the detector output operates a modulator, which controls the frequency of the receiver local oscillator, in such a way that the frequency deviation of the received wave is reduced before the wave is amplified at the intermediate frequency. This arrangement was first described by Chaffee.[6]

[6] J. G. Chaffee, "The application of negative feedback to frequency modulation systems," *Bell Sys. Tech. Jour.*, vol. 18, pp. 404–438; July, 1939.

In what follows it is supposed that the detector output is equal to the frequency deviation of the wave applied to it, and that the modulator is also free from distortion. In practice, it is the modulator distortion which sets the limit to reduction in distortion obtainable by feedback.

Let the frequency deviation of the received wave be $\Delta\omega_1 S(t)$, and let the detector output be $O(t)$. The frequency deviation of the local oscillator is $\beta O(t)$, where β is a constant, and the frequency deviation of the wave of intermediate frequency is $\Delta\omega_1 S(t) - \beta O(t)$. The effect of the i.f. amplifier is to delay the modulation by a time T, and to add distortion, so that the frequency deviation of the wave arriving at the detector, and hence the detector output, is $\Delta\omega_1 S(t-T) - \beta O(t-T) + KD(t-T) = O(t)$. $D(t-T)$ is the relative distortion in the frequency deviation, and K is equal to the maximum deviation at intermediate frequency.

By expanding $O(t-T)$ as a series of derivatives of $O(t)$, an equation is obtained expressing $O(t)$ in terms of $S(t-T)$, $D(t-T)$, and derivatives of $O(t)$. On differentiating this equation and eliminating $O'(t)$ between this equation and the first one, a new equation is obtained from which $O'(t)$ is absent. Proceeding in this way, all the derivatives of $O(t)$ can be eliminated and the other terms collected together to give

$$O(t) = \frac{\Delta\omega_1}{1+\beta}\left[S(t - T/(1+\beta)) + \frac{1}{1+\beta} D(t - T/(1+\beta))\right]$$

$$+ \text{terms of higher order.}$$

The higher-order terms are derivatives of $S(t)$ and $D(t)$ and are always of negligible magnitude. The term $D(t-T/(1+\beta))$ represents the distortion suffered by a wave whose frequency deviation is $\Delta\omega S(t-T/(1+\beta))$ where $\Delta\omega = \Delta\omega_1/(1+\beta)$. Under different conditions which will now be examined, the magnitude of

$$\frac{1}{1+\beta} D(t - T/(1+\beta))$$

is more or less changed when β is varied.

First, let the bandwidth remain constant as β is increased from zero. Then, from (7), (8), (12), and (13), the quadratic distortion (S^2, SS' etc.) is proportional to $\Delta\omega/(1+\beta)$, i.e., to $(1+\beta)^{-2}$, the cubic distortion to $(1+\beta)^{-3}$, and generally the nth order distortion is proportional to $(1+\beta)^{-n}$. Thus, if the application of feedback reduces the deviation of the received wave by N decibels, the quadratic distortion is reduced by $2N$ decibels and the cubic distortion by $3N$ decibels.

Next, suppose that, as the maximum deviation is reduced by feedback, the bandwidth of the i.f. amplifier is reduced in the same ratio, the shape of the transfer characteristic being kept constant. This is possible so long as the reduced bandwidth is greater than twice the highest modulating frequency. From (5) it is seen that the distortion of all orders due mainly to the phase characteristic remains constant, and the distortion due mainly to the amplitude characteristic increases in the ratio $1+\beta$.

The third case to be considered is that in which the deviation ratio is initially large, and the feedback is sufficiently great to reduce the deviation ratio to less than 1. As feedback is applied the bandwidth is reduced from its initial value of $2\Delta\omega_1$ to the limiting value $2\omega_q$. The distortions before and after feedback are not directly comparable on a numerical basis, since the nature of the distortion changes. However, in some cases the distortion for both large and small deviation ratios is due mainly to one particular term. As an example of typical distortion components, the term

$$\frac{1}{\Delta\omega}\cdot\frac{d}{dt}\cdot\frac{\phi_3}{3!}\left(\frac{\Delta\omega S}{\omega_B}\right)^3$$

in (8) and the corresponding term

$$\frac{1}{2}\,\frac{\Delta\omega^2}{\omega_B{}^2}\,\frac{d}{dt} S^3(1/3Q_3 - Q_2P_1)$$

in (13) may be taken. From the relations between the polar and Cartesian forms of the transfer characteristic given in the Appendix, these two terms are equal.

Initially ($\beta = 0$ $\omega_B = \Delta\omega_1$), the distortion is

$$\frac{\phi_3}{2}\,\frac{S^2S'}{\Delta\omega_1},$$

and finally ($\omega_B = \omega_q$), it is

$$\frac{\phi_3}{2}\,\frac{\Delta\omega_1{}^2}{\omega_q{}^3}\,\frac{S^2S'}{(1+\beta)^3}.$$

The distortion is therefore reduced in the ratio

$$\left\{\frac{\Delta\omega_1}{\omega_q(1+\beta)}\right\}^3 = [D/(1+\beta)]^3.$$

Appendix

A. Relations between the polar and Cartesian forms of the transfer characteristic

The steady-state transfer characteristic $T(u)$ may be written $T(u) = exp\{A(u) + j\phi(u)\} = P(u) + jQ(u)$, where $A(u)$ is the amplitude characteristic in nepers, $\phi(u)$ the phase characteristic in radians, and $P(u)$ and $Q(u)$ are the in-phase and quadrature components of the characteristic.

Then

$$P(u) = \exp A(u) \cos \phi(u)$$

$$Q(u) = \exp A(u) \sin \phi(u)$$

$$A(u) = 1/2 \log h[P^2(u) + Q^2(u)]$$

$$\phi(u) = \tan^{-1} [Q(u)/P(u)].$$

It is assumed that $A(u)$, $\phi(u)$, $P(u)$, and $Q(u)$ may be expressed as power series of the form

$$A(u) = A_0 + uA_1 + \frac{u^2}{2!}A_2 + \frac{u^3}{3!}A_3 + \cdots.$$

The coefficient A_0 represents the gain of the network at the reference frequency. Since this is arbitrary, it is convenient to put $A_0=0$. The coefficient ϕ_0 represents a constant phase change of the carrier wave, which is of no interest, and ϕ_1 represents a time delay of the modulation impressed on the carrier which may be allowed for, if necessary, in the final result.

On the assumption that $A_0=\phi_0=\phi_1=0$, the relations between the coefficients in the power series for $A(u)$, $\phi(u)$, $P(u)$, and $Q(u)$, are as follows:

$$P_0 = 1$$
$$P_1 = A_1$$
$$P_2 = A_2 + A_1^2$$
$$P_3 = A_3 + 3A_2A_1 + A_1^3$$
$$P_4 = A_4 + 4A_3A_1 + 3A_2^2 + 6A_2A_1^2 + A_1^4 - 3\phi_2^2$$
$$P_5 = A_5 + 5A_4A_1 + 10A_3A_2 + 10A_3A_1^2 + 15A_2^2A_1 + 10A_2A_1^3 + A_1^5 - 15A_1\phi_2^2 - 10\phi_3\phi_2$$

$$Q_0 = Q_1 = 0$$
$$Q_2 = \phi_2$$
$$Q_3 = \phi_3 + 3\phi_3A_1$$
$$Q_4 = \phi_4 + 4\phi_3A_1 + 6\phi_2A_2 + 6\phi_2A_1^2$$
$$Q_5 = \phi_5 + 5\phi_4A_1 + 10\phi_3A_2 + 10\phi_3A_1^2 + 10\phi_2A_3 + 30\phi_2A_2A_1 + 10\phi_2A_1^3$$

$$A_1 = P_1$$
$$A_2 = P_2 - P_1^2$$
$$A_3 = P_3 - 3P_2P_1 + 2P_1^3$$
$$A_4 = P_4 - 4P_3P_1 - 3P_2^2 + 12P_2P_1^2 - 6P_1^4 + 3Q_2^2$$
$$A_5 = P_5 - 5P_4P_1 - 10P_3P_2 + 20P_3P_1^2 + 30P_2^2P_1 - 60P_2P_1^3 + 24P_1^5 - 30P_1Q_2^2 + 10Q_3Q_2$$

$$\phi_2 = Q_2$$
$$\phi_3 = Q_3 - 3Q_2P_1$$
$$\phi_4 = Q_4 - 4Q_3P_1 - 6Q_2P_2 + 12Q_2P_1^2$$
$$\phi_5 = Q_5 - 5Q_4P_1 - 10Q_3P_2 + 20Q_3P_1^2 - 10Q_2P_3 + 60Q_2P_2P_1 - 60Q_2P_1^3.$$

B. Calculation of harmonics and intermodulation products

If the modulating wave is comprised of two cosine waves of different amplitudes, then in calculating distortion products it is necessary to expand expressions of the form $(k_1 \cos\theta_1 + k_2 \cos\theta_2)^n$ in terms of unit powers of multiple angles, n being a positive integer. This can be done most conveniently by expressing the cosine terms in exponential form and applying the multinomial theorem to expand the result.

Thus, if

$$y = [k_1 \cos\theta_1 + k_2 \cos\theta_2]^n$$
$$= \frac{1}{2^n}[k_1 \exp j\theta_1 + k_1 \exp - j\theta_1 + k_2 \exp j\theta_2 + k_2 \exp - j\theta_2]^n,$$

then, by the multinomial theorem,

$$y = \frac{n!}{2^n}\times \sum \frac{k_1^{(\alpha_1+\alpha_2)}k_2^{(\alpha_3+\alpha_4)} \exp j\{(\alpha_1-\alpha_2)\theta_1+(\alpha_3-\alpha_4)\theta_2\}}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!}. \quad (18)$$

α_1, α_2, α_3 and α_4 are positive integers or zero, and the summation extends over all possible values of α_1, α_2, α_3, and α_4 consistent with the relation

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = n. \quad (19)$$

The coefficient of $\frac{1}{2} \exp j(p\theta_1+q\theta_2)$, which is the same as that of $\cos(p\theta_1+q\theta_2)$, is obtained by putting $\alpha_1-\alpha_2=p$, $\alpha_3-\alpha_4=q$. The coefficient of $\cos(p\theta_1-q\theta_2)$ is the same as that of $\cos(p\theta_1+q\theta_2)$, since it can be formed simply by interchanging the values of α_3 and α_4.

The minimum value of α_2 and α_4 is zero. Let $\alpha_2=r$ then

$$\alpha_1 = p + r \quad \alpha_3 = \frac{n-p+q}{2} - r \quad \alpha_4 = \frac{n-p-q}{2} - r.$$

It is clear that r attains a maximum value when $\alpha_4=0$ and this maximum is $\frac{1}{2}(n-p-q)$. Substituting for α_1, α_2, α_3, and α_4, and summing over all possible values of r, the coefficient of $\cos(p\theta_1 \pm q\theta_2)$ given by (18) is

$$\frac{n!}{2^{n-1}}\sum_{r=0}^{1/2(n-p-q)} \frac{k_1^{(p+2r)}\,k_2^{(n-p-2r)}}{r!(p+r)!\left(\frac{n-p-q}{2}-r\right)!\left(\frac{n-p+q}{2}-r\right)!} \quad (20)$$

Only certain values can be taken by p and q. If n is even, $p+q$ must be even, and if n is odd, $p+q$ must be odd. Also, $p+q$ cannot be greater than n. If either k_1 or k_2 is zero, the expression has a value only when either $p+2r=0$ or $n-p-2r=0$. If n is even, y has a mean value which is $\frac{1}{2}$ of the value found by putting $p=q=0$ in (20).

C. Measurement of intermodulation distortion

It was shown in Section VII that, if a carrier wave modulated by a wave $C_1 \cos\omega_1 t + C_2 \cos\omega_2 t$ ($C_2 \ll C_1$ $\omega_1 \ll \omega_2$) is passed through a network having a nonlinear phase characteristic, the h.f. component, $C_2 \cos\omega_2 t$, of the modulating wave becomes modulated in frequency by a function of the l.f. component. The modulated high-frequency component is, from (14),

$$C_2 \cos \left\{\omega_2 t + \frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right)\right\}. \qquad (21)$$

The intermodulation distortion is defined as the maximum variation of the quantity

$$1 + \frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right)$$

from its mean value.

Suppose that the modulated carrier emerging from the network is applied to an ideal detector which yields an output proportional to the frequency deviation. From this output the component given by (21) is selected by means of a suitable filter and applied to a differentiating circuit (e.g., a series RC circuit of small time constant with the output taken across R) to produce a wave proportional to

$$-C_2\omega_2\left\{1 - \frac{\Delta\omega\omega_1}{\omega_B^2}\sin\omega_1 t\,\phi''\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right)\right\} \cdot \sin\left\{\omega_2 t + \frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right)\right\}.$$

This wave is applied to an amplitude detector which produces an output proportional to

$$\omega_2\left\{1 - \frac{\Delta\omega\omega_1}{\omega_B^2}\sin\omega_1 t\,\phi''\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right)\right\}.$$

The alternating part of the output is filtered out and applied to an integrating circuit (a series RC circuit of large time constant with the output taken across C) to give an output proportional to

$$\frac{\omega_2}{\omega_B}\phi'\left(\frac{\Delta\omega}{\omega_B}\cos\omega_1 t\right),$$

the peak value of which may be mesaured by a vacuum-tube voltmeter. The voltmeter may be calibrated to read directly the intermodulation distortion.

A Study of Tropospheric Reception At 42.8 Mc. and Meteorological Conditions*

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Summary—**From February, 1945, during its hours of operation, station W2XMN at Alpine, N. J., has been recorded at Needham, Mass., at a distance of 167 miles. W2XMN operates on a frequency of 42.8 Mc., at a power of 50 kw., and its daily schedule is from 1600 to 2300, E.D.S.T. in summer and E.S.T. in winter. Analysis of the Alpine recording has shown that no part of the ionosphere is involved in the transmission, which is purely tropospheric.**

The Alpine fields show a marked seasonal change, being much higher in the summer than in the winter, and this has been found to be principally due to the seasonal changes in surface refraction along the transmission path. A controlling factor in the seasonal change of refraction is water-vapor pressure, which is at a maximum in the summer.

All types of frontal passage are found to lower transmission, and, presumably because of wave-guide effects, the amount of field depression caused by the passage of the front varies with the angle made by the front with the path. When the front is parallel with the path, the field is least depressed, but is lowest when the front makes a considerable angle with the path.

High fields at Needham are usually followed by an increase in surface temperature along the path, the temperature reaching a maximum about 30 hours after the field maximum. Conversely, low fields are generally followed by falling temperatures, which reach a minimum some 30 hours after the field minimum.

The best transmission along the Alpine-Needham path occurs when the wind velocity on the path is lowest, and the worst transmission accompanies high winds, probably because of turbulence which breaks up favorable stratification in the lower atmosphere.

Finally, the direction of air movement with respect to the path is related to transmission, Needham fields being higher when the wind is parallel with the path. The principal conditions favorable for transmission over this path are therefore summer, high surface refraction, rising temperatures, low wind velocities, winds parallel with the path, and an absence of frontal passages.

Introduction

TRANSMISSION from the f.m. station W2XMN, operating on 42.8 Mc., at Alpine, N. J., is received at Needham, Mass., distant 167 miles, on a half-wave dipole with reflector 50 feet above ground. In a conventional receiver a variable diode load is utilized to operate a Micromax single-pen recorder. The circuit is periodically calibrated with a Ferris Microvolter at 42.8 Mc. when W2XMN is off the air. Because of the large fading amplitudes, the fields are transcribed from the recorder charts as log microvolts at the receiver. One microvolt at the receiver equals approximately 0.7 microvolt per meter at the antenna. The charts are scaled for the median value of 20-minute intervals from which are derived hourly means and the mean nightly field for the seven hours during which Alpine is on the air each day. In the day-by-day comparisons given below with the tropospheric elements, a general use has been made of the ratio of the daily log field to a

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