The Theory of Impulse Noise in Ideal Frequency-Modulation Receivers*

DAVID B. SMITH†, SENIOR MEMBER, I.R.E., AND WILLIAM E. BRADLEY†, SENIOR MEMBER, I.R.E.

Summary—**The following paper contains a quantitative analysis of the effect of impulse noise on ideal frequency-modulation receivers. It is shown that two types of detected noise may result from an impulse transient. The amplitude and wave form of the generated noise are substantially independent of the amplitude or wave form of the initiating noise provided the noise transient exceeds the desired signal. Of the two types, the weaker is determined largely by the characteristics of the audio amplifier and results from a perturbation of the phase of the detector signal by the noise. The characteristics of the second and more objectionable type are established by the de-emphasis circuit and result when the phase of the detector signal is caused to slip one revolution by the noise. The question as to which type of noise will obtain is shown to be purely a matter of chance. An operational formula for the ideal detection process is also given from which both steady-state and transient solutions of the process of detection may be derived.**

Introduction

UNDER ACTUAL listening conditions, with present frequency-modulation broadcast receivers, thermal noise is not noticeable except under extremely poor receiving conditions. On the contrary, noise picked up from other electrical apparatus, such as automobile ignition systems, sparking commutators, or relay contacts, is the limiting factor and is audible ordinarily as isolated clicks or pops standing out against a relatively quiet background. Because of the prevalence of this type of noise under practical reception conditions, this report is concerned with the behavior of a frequency-modulation receiver when a noise impulse is applied to its input while it is receiving a constant-amplitude carrier wave which is frequency modulated with an audio program. The impulse noise in accordance with present experience is taken to be of very short time duration, delivering its energy to the first tuned circuit of the receiver in a time short compared to the time-constant of that circuit.

Analysis of Receiver Operation

It is very difficult to carry through an exact analysis of the behavior of a receiver under all conditions. However, by making certain simplifying assumptions a fairly accurate and very illuminating picture of the operation can be obtained. The simplified receiver is defined to consist of a linear band-pass filter, an ideal frequency-modulation detector including de-emphasis[1] and a linear low-pass audio amplifier. In general, when such a receiver is excited by an impulse noise, a transient is produced which will exceed the desired signal and momentarily take control of the detector causing a perturbation of the desired detected signal. In our analysis we shall first determine the nature of the transient signal produced at the detector input as a result of impulse-noise excitation of the linear band-pass filter; second, the form of the combined noise and desired signal at the detector input; and third, the process of detection and the audio signal resulting from detection of the combined signals.[2]

There is, of course, a variety of specific forms of band-pass filters which might be used in a frequency-modulation receiver. All of them, however, will be characterized in a practical set by having adequate bandwidth to transmit the desired frequency-modulated signal with as much attenuation outside the channel as can conveniently be obtained. It is characteristic of such an amplifier that when shock-excited it will ring or oscillate at its natural frequency producing a wave train which builds up and decays at a rate determined by the bandwidth of the amplifier. Variations in the specific form of the noise impulse and variations of pole configurations of the amplifier will vary the exact form of the noise envelope, but in general these variations are all minor. To get a quantitative picture of the behavior of the amplifier, we will analyze a system of n identical single-tuned stages; however, were other practical amplifiers employed, the results from a noise standpoint would not be essentially different. The operational form for the response of such an amplifier is given by

$$e_n(t) = \left[\frac{g_m}{c}\right]^n \left[\frac{p}{(p+\alpha)^2+\omega_0^2}\right]^n \cdot e_0(t) \qquad (1)$$

in which n is the number of stages, α is the damping factor of each stage, ω_1 is the natural angular frequency, p is the usual Heaviside operator, and g_m is the interstage transconductance. The initial driving current is $e_0 \cdot g_m$ and $\omega_0^2 = \omega_1^2 - \alpha^2$. For steady-state conditions the frequency-response curve of such a filter will have a moderately blunt nose with steep-sloping sides. To complete the synthesis of the amplifier we will so adjust the individual stage damping that at frequencies off the mid-band point by an amount δ_0 equal to the deviation of the frequency-modulated signal for 100 per cent modulation, the over-all response of the filter will be down 3 decibels. As shown in Appendix I, which contains a detailed analysis of the filter for steady-state and

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† Philco Corporation, Philadelphia, Pennsylvania.

[1] The I.R.E. Technical Committee on Frequency Modulation has proposed the following definition of an ideal frequency-modulation detector: "1FM27. *Ideal Frequency-Modulation Detector.* A detector whose voltage or current output is proportional to the frequency deviation of a modulated wave and which is unresponsive to amplitude modulation." We have modified it only to the extent of including the de-emphasis circuit as part of the detector.

[2] V. D. Landon, "Impulse noise in frequency-modulation reception," *Electronics*, vol. 14, pp. 26–76; February, 1941.

noise conditions, this bandwidth obtains when

$$\alpha = \frac{\delta_0}{\{2^{1/n} - 1\}^{1/2}}. \qquad (2)$$

Values of this ratio for various values of n are given in Table I. As the carrier frequency ω_0 can be any convenient value $\gg \delta_0$, this completes the synthesis of the filter.

In addition, it is convenient to use $1/\delta_0$ as the unit of time against which to observe the variations and interplay of the several signals, and this will be done throughout the rest of the paper. As δ_0 is in terms of angular velocity, $1/\delta_0$ is the time required for a sinusoidal signal corresponding in frequency to the deviation for 100 per cent modulation to advance in phase by one radian.

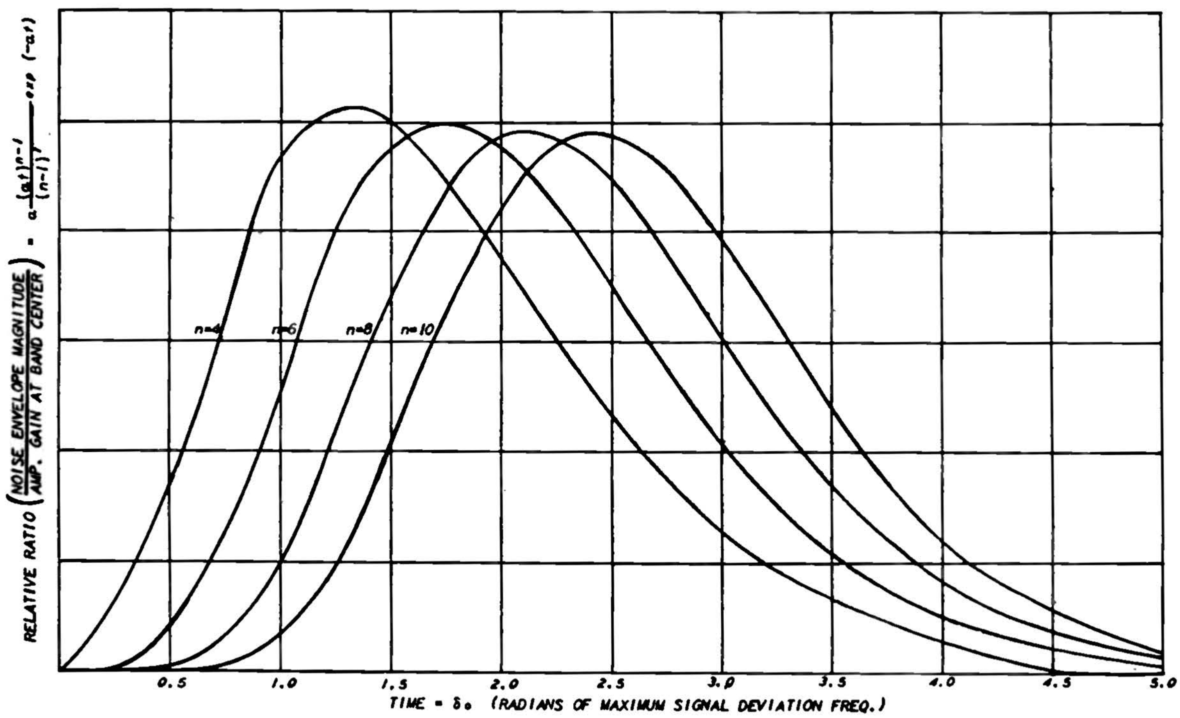

Fig. 1 —Noise-pulse envelopes for intermediate-frequency amplifiers of number of stages n having identical bandwidth over all, between half-power points.

Noise-pulse envelope form $N(t) = a \dfrac{(at)^{n-1}}{(n-1)^1} \exp(-at)$

where $a = \dfrac{\delta_0}{\{2\ 1/n - 1\}^{1/2}}$ (to make over-all bandwidth constant for any n).

The Technical Committee on Frequency-Modulation Receivers of the Institute of Radio Engineers has tentatively defined impulse noise as follows: "1FM38. *Impulse Noise.* Noise characterized by transient disturbances separated in time by quiescent intervals." In practice it has been observed that the time duration of most common impulse-noise sources such as ignition systems, door bells, etc., is less than $1/\delta_0$. This can be verified by observing such noise on a wide-band system such as a television receiver. Under these circumstances, the exact wave shape is not important and has little to do with the resulting transient in the filter. The characteristic which is important is the area under the initial impulse-voltage-envelope time curve, which area we shall call D. For analytical purposes, any short exciting impulse of area D and duration less than $1/\delta_0$ can be represented by the spike function $Dp1$.[3] As shown in Appendix I when the n-stage filter is excited by this function it produces a transient noise signal

TABLE I

$$\frac{\delta_0}{\alpha} = \{2^{1/n} - 1\}^{1/2}$$

n	δ_0/α
1	1.000
2	0.644
3	0.510
4	0.435
5	0.385
6	0.350
7	0.323
8	0.302
9	0.283
10	0.268
11	0.255
12	0.244

$$N_n(t) \cong 2D \cdot \alpha \cdot \left[\frac{g_m}{G}\right]^n \cdot \frac{(\alpha t)^{n-1}}{(n-1)!} \cdot \exp(-\alpha t) \cdot \cos \omega_0 t. \qquad (3)$$

[3] This tacitly assumes the nominal carrier frequency of the noise train to be the same as that of the filter. If it is not, D will be reduced by an amount determined by the relative energy content at the band-pass frequencies.

This corresponds to a signal having a fixed frequency about equal to that of the unmodulated program signal carrier and having an envelope which rises to a peak amplitude at $\alpha t = n-1$ and then decays more or less exponentially. Some of these envelopes for various values of n are shown in Fig. 1. Since α is uniquely determined by the half bandwidth δ_0, the resulting transient is largely determined by the characteristics of the receiver. Hence, when a noise impulse occurs which produces a transient greater than the program signal, the noise momentarily "captures" the detector and takes over the control of the detected signal. When such a "capture" happens we are first interested in how long it lasts, and second in what effect it produces in the detector. Neglecting the minor variation in amplitude of the desired program signal as a result of modulation and the frequency-response characteristic of the filter, we find that the ratio of the peak noise envelope to the program signal envelope at the detector is given by

$$\frac{e^{N}{}_{n\max}}{E_{n\max}} = \frac{2\cdot D\cdot \delta_0}{E_0}\left\{\frac{(n-1)^{n-1}\cdot \exp(1-n)}{\{2^{1/n}-1\}^{1/2}\cdot (n-1)!}\right\} = S. \quad (4)$$

This expression, while very formidable looking, is nevertheless practically independent of n for three or more stages. The function of n is tabulated in Table II. It is interesting to note that, for a given disturbance and

TABLE II

$$A = \frac{(n-1)^{n-1}\exp(1-n)}{\{2^{1/n}-1\}^{1/2}(n-1)!}$$

n	A
1	1.00
2	0.57
3	0.53
4	0.51
5	0.51
6	0.50
7	0.50
8	0.50
9	0.49
10	0.49
11	0.49
12	0.49

desired signal, the resulting transient noise to signal is directly proportional to the receiver bandwidth.

The capture time can be determined as shown in the

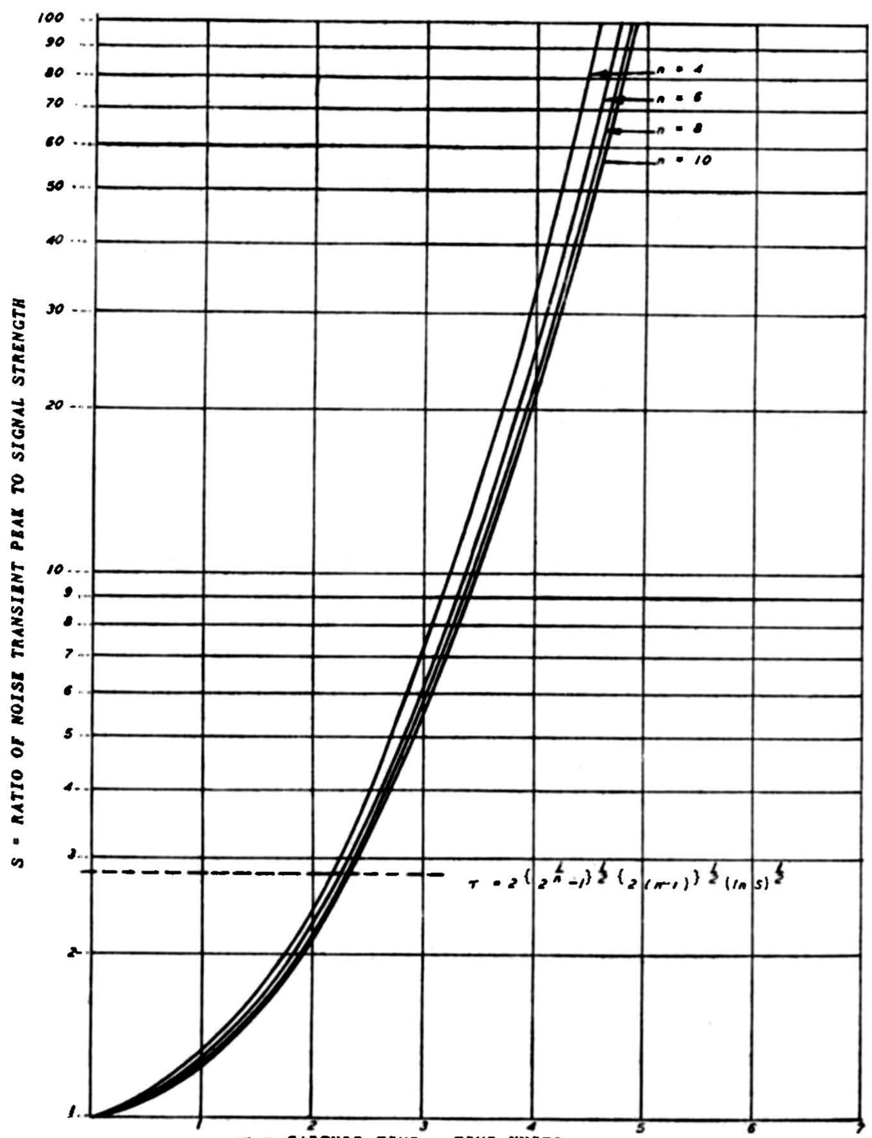

Fig. 2—Effect of noise-to-signal ratio on capture time. One time unit = 1 radian of maximum deviation frequency.

appendix by calculating the time interval during which the noise envelope exceeds that of the program signal at the detector. This time, which we shall call τ, is approximately given by

$$\tau \cong 2\cdot\{2^{1/n} - 1\}^{1/2}\cdot\{2(n-1)\}^{1/2}\{\ln S\}^{1/2}. \qquad (5)$$

Again the expression, while formidable in appearance, is practically independent of n and varies only slightly with the noise transient to signal ratio S. The value of τ for various values of n and S is given in Fig. 2. In general τ is of the order of three or four. Essentially, this means that if the noise impulse at the detector is not as strong as the signal there is no serious interruption, but if the noise once captures the signal, then it does not make too much difference whether it is a strong noise or a weak one.

While we are nominally dealing with a frequency-modulation system, it is more convenient to deal in terms of the phase variation since this is tangible and can be easily visualized. During the admixture of noise transient and desired signal, certain rather striking phenomena take place. Before discussing this, however, we need an operational expression for the detection process.

We shall define $\phi(t)$ as the phase departure of the modulated program carrier from its unmodulated value. Then the radio-frequency carrier is given by

$$E_{\text{radio frequency}} = E\cdot\cos\{\omega_0 t + \phi(t) + \theta\} \qquad (6)$$

where θ is an arbitrary radio-frequency phase angle as observed at the detector input. The transmitter contains some device such as a Travis modulator which produces a frequency deviation of the carrier in proportion to the strength of the modulating program signal. If h is the conversion constant relating frequency deviation to modulating signal strength, then in operational terms

$$\phi'(t) = \frac{1}{p}\cdot h\cdot e_p(t) \qquad (7)$$

where $e_p(t)$ is the program signal and the prime on $\phi'(t)$ indicates the lack of pre-emphasis. If pre-emphasis with a time constant $1/\gamma$ is employed, then the operational form including the effect of pre-emphasis is given by

$$\phi(t) = \left[\frac{p+\gamma}{p\cdot\gamma}\right]\cdot h\cdot e_p(t). \qquad (8)$$

For a steady-state condition, p may be replaced by $j\sigma$, where σ is the frequency of the program signal. Hence, for steady-state monotone conditions

$$\phi(t) = Re\left\{\left[\frac{j\sigma+\gamma}{j\sigma\gamma}\right]\cdot h\cdot e_p\cdot\exp j\sigma t\right\}.$$

If the receiver is ideally frequency responsive, that is, responds only to the phase angle and not to amplitude, then the reverse of (8) takes place in the process of reception. If we assume a reciprocal frequency-conversion constant and neglect variations in level, then the audio output signal e_d including de-emphasis is related to $\phi(t)$ by the operational form

$$e_d(t) = \left[\frac{p\cdot\gamma}{p+\gamma}\right]\cdot\frac{1}{h}\cdot\phi(t) \qquad (9a)$$

and

$$e_d(t) = \left[\frac{p+\gamma}{p\cdot\gamma}\right]\cdot h\cdot\left[\frac{p\cdot\gamma}{p+\gamma}\right]\frac{1}{h}\cdot e_p(t) = e_p(t). \qquad (9b)$$

The above equations completely define the mechanism of modulation under ideal conditions for both transient and steady-state conditions. Further, they show that, when pre-emphasis is included, the output signal after de-emphasis is that signal which would be obtained across the resistance of a series resistance-capacitance circuit having the de-emphasis time constant and driven by a voltage proportional to the phase deviation of the carrier. Such a circuit will, of course, attenuate frequencies below the transition frequency of the de-emphasis network; consequently, $\phi(t)$ must be quite large for low-frequency signals.

On a vector diagram it is convenient to take the center frequency as the reference, and in this case $\phi(t)$ is represented by a spoke of constant amplitude with its hub at the origin deviating to and fro from the reference axis. During the course of one cycle of low-frequency program signal it may wind up many revolutions before it reverses and goes back. The effect of mistuning the receiver may be included by adding a term βt to $\phi(t)$, in which β is the angular frequency difference between actual tuning and the proper tuning. This superimposes on $\phi(t)$ a slow constant rate of rotation β in the direction determined by the sign of β.

Finally, we need to know the maximum rate of change of $\phi(t)$ in terms of our time unit $1/\delta_0$, and this follows from our definition of δ_0. The rate of change of $\phi(t)$ is directly proportional to the instantaneous degree of modulation and at 100 per cent modulation is numerically equal to one radian per time unit.

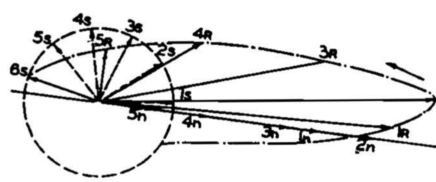

Fig. 3—Vector diagram showing how noise pulse causes momentary change of resultant phase, resulting in a click.

When noise and signal occur simultaneously at the detector input, it is the phase of the resultant of the two wave trains which must be used to calculate $e_d(t)$ the audio output. This phase angle as a function of time may be visualized on a complex number diagram as shown in Fig. 3, on which waves of angular frequency ω_0 are stationary. The dotted circle near the origin represents the level of the program signal. In order that the diagram represent a typical situation, it is assumed that the desired signal is deviated somewhat from ω_0 and hence is represented by a slowly rotating vector. Several successive positions of the signal in the absence of noise are depicted by the numbers 1*s* to 6*s* on Fig. 3.

When a noise transient occurs, it may be regarded as a fixed vector which grows out from the hub, past the circle to some maximum value, and then recedes. A possible set of positions of the noise vector for the same times are given by the numbers *1n* to *6n*. The time that it exceeds the circle is given by τ in (5) above. This may amount to three or four time units and, with maximum modulation, the signal vector may rotate three or four radians during this interval.

The successive positions of the terminus of the resultant of the program signal plus the noise transient are also shown by the numbers *1r* to *6r*. It can be seen that, qualitatively speaking, the phase is momentarily perturbed by the noise but shortly resumes its former locus. The consequences at the audio output of this momentary excursion of phase are treated in detail later, but it is apparent that, since the duration of the whole excursion is of the order of magnitude of the reciprocal of the total receiver bandwidth, and since the amplitude of the excursion is limited to less than π radians, the effect at the audio output is small if not insignificant for the case shown in Fig. 3, and sounds somewhat like a faint click.

A startlingly different result appears if, by chance, the relative phase of the noise-wave train and the useful signal have the relations shown in Fig. 4. In this case, the resultant phase makes a backward loop around the origin ending up with a permanent displacement of one whole revolution. After this sudden discontinuity, the phase is again under the control of the useful signal and continues its prescribed course as if nothing had happened. In other words, in the first case, the phase of the

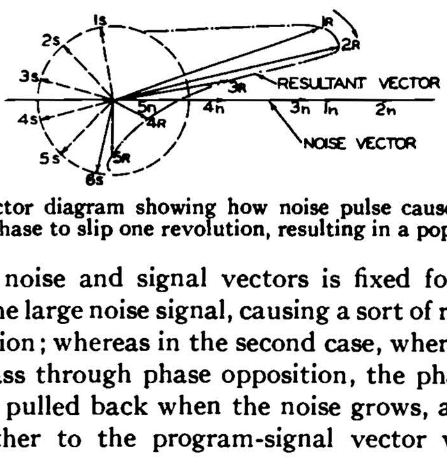

Fig. 4—Vector diagram showing how noise pulse causes resultant phase to slip one revolution, resulting in a pop.

resultant noise and signal vectors is fixed for a short time by the large noise signal, causing a sort of rectilinear perturbation; whereas in the second case, where the two signals pass through phase opposition, the phase of resultant is pulled back when the noise grows, and drops back further to the program-signal vector when the noise transient is over, having in the process slipped one complete revolution.

Due to the fact that, in a relatively short time, the phase of the resultant undergoes a permanent displacement, the effect on the audio output can be investigated by assuming that $\phi(t)$ has subtracted from it a noise impulse in the form of a unit function. In this case, the audible effect is quite noticeable and sounds like a "pop."

A very interesting feature of this type of interference is the fact that whether a radio-frequency noise impulse of sufficient amplitude produces a "pop" or a faint "click" depends only on the radio-frequency phase relation between signal and noise transient. Since this phase is purely random, quantitative calculations of noise power output must be done on the basis of probability.

A little thought shows that if, during the time that the noise train exceeds the signal, the two trains pass through phase opposition, then the phase undergoes the permanent displacement of one revolution in the direction towards reducing the phase deviation. Since all relative phases are equally likely, the probability that the signal and noise will pass through phase opposition during the capture time is numerically equal to the fraction of a revolution that the useful signal changes in phase during this time. If k is the ratio of the instantaneous deviation (at the time of the noise impulse) to the deviation corresponding to 100 per cent modulation, then the phase of the program signal will change by $(k+\beta/\delta_0)\tau$ radians during the capture time.

The factor β/δ_0 represents effect of mistuning and for low modulation is the predominant factor. Hence, the probability P that a sufficiently strong noise impulse will produce a pop noise is given by

$$P = (k + \beta/\delta_0)\cdot\frac{\tau}{2\pi}\cdot \tag{10}$$

If the average rate of noise impulses of sufficient strength to capture the desired signal is given by R, then it is to be expected that the average rate of pop noises will be RP and the average rate of click noises, some of which may be inaudible, will be $R(1-P)$. It is also possible that in a few instances the program signal and noise might (with high modulation) pass through phase opposition twice. In this case the permanent phase displacement would be twice as large, but otherwise the same.

Detected Signal-to-Noise Ratio

With the above expression we can now calculate the detected signal-to-noise ratio in terms of probabilities, for noises of the pop type. When the phase of the resultant vector at the detector input slips one revolution, the effect is to superimpose on $\phi(t)$ an additional term having the form of a unit function of amplitude 2π. Hence, the detected signal becomes[4]

$$e_d(t) = \left[\frac{p\cdot\gamma}{p+\gamma}\right]\frac{1}{h}\left[\phi(t) \pm 2\pi 1\right]. \tag{11}$$

Performing the indicated operations, we find then that there is a noise term $e_{dn1}(t)$ subtracted from the detected useful signal of the following form:

$$e_{dn1}(t) = 2\pi\gamma\cdot h \exp(-\gamma t) \tag{12a}$$

or since E_{dm}, the detected signal for 100 per cent modulation, is equal to $h\delta_0$ we can rewrite (12a) as

$$e_{dn1}(t) = \left(\frac{2\pi\gamma}{\delta_0}\right)\cdot E_{dm}\cdot\exp(-\gamma t). \tag{12b}$$

[4] The polarity of the unit function is always opposite the polarity of the time derivative of $\phi(t)$ at the time of the noise impulse.

The sign of the noise signal is such as momentarily to reduce the amplitude of the detected signal, and the wave form is simply the discharge curve of a resistance-capacitance circuit having the de-emphasis time constant. For a system of 75-kilocycle deviation for maximum modulation and a time constant of 100 microseconds, the peak amplitude of the generated noise signal is about 13 per cent of the program signal amplitude corresponding to 100 per cent modulation.

The output power for this type of noise is the product of the number of input pulses per second, the probability of a pop noise occurring, and the integral of the square of the voltage of a pop noise pulse. In calculating the probability it must be kept in mind that this is a function of the instantaneous modulation. For example, if sinusoidal modulation is employed, the probability (neglecting mistuning) of a noise pulse occurring is zero when the deviation is zero and a maximum for the peak of modulation. Hence,

$$\frac{W_{n1}}{W_{s\max}}=\frac{R\cdot\tau}{2\pi}\,\frac{\left\{\dfrac{1}{\dfrac{\pi}{\sigma}}\displaystyle\int_{-\pi/2\sigma}^{+\pi/2\sigma}\left(k\cdot\cos\sigma t+\frac{\beta}{\delta_0}\right)dt\right\}\left\{\left(\dfrac{2\pi\gamma}{\delta_0}\right)^2\displaystyle\int_0^\infty\exp(-2\gamma t)dt\right\}}{\dfrac{1}{\dfrac{\pi}{\sigma}}\displaystyle\int_{-\pi/2\sigma}^{\pi/2\sigma}\{\cos\sigma t\}^2dt}$$

$$=\frac{2\pi R\cdot\tau\gamma}{\delta_0^2}\left\{\frac{2k}{\pi}+\frac{\beta}{\delta_0}\right\}\qquad(13)$$

where W_{n1} and $W_{s\max}$ represent the probable noise power due to pops and the desired audio signal power for 100 per cent modulation, respectively.

In the case of the click type of noise mentioned above, the amplitude of the resultant noise is considerably less for two reasons. In the first place, the maximum possible phase excursion is $\pm\pi$ radian, and second, the duration of the excursion is very short instead of being permanent. From the above discussion it will be seen that there is equal probability of a click noise having any value of phase displacement from $-\pi$ to $+\pi$ radians. Hence, we will call the amplitude of the displacement $Q\pi$, where Q is a probability factor and has an equal chance of being any value from -1 to $+1$.

The duration of the excursion is equal to the capture time τ. Hence, for this type of noise the program signal has added to it a noise impulse in the form of a sudden excursion of amplitude $Q\pi$ lasting for a time interval τ. As before, the resultant detected noise can be found by applying the detector operator. Hence, with click noise the signal plus noise is given by

$$e_d(t)=\left[\frac{p\cdot\gamma}{p+\gamma}\right]\frac{1}{h}\left[\phi(t)+\pi Q(1-1_\tau)\right]\qquad(14)$$

where, as is customary, the subscript τ on the second unit function indicates that the step function starts at $\delta_0 t=\tau$ instead of $t=0$. The noise part of the signal E_{dn2} which in phase represents a rectangular pulse of duration τ will, of course, pass through the circuit represented by the operation substantially without change, since τ/δ_0 is small as compared with $1/\gamma$. It will, however, be attenuated and integrated by the audio circuit if the bandwidth of the circuit is appreciably less than δ_0. Let us suppose that the audio amplifier has a cutoff as determined by a resistance-capacitance circuit of bandwith σ_0 radians per second and that $\sigma_0\ll\delta_0$. The transient resulting in such an amplifier can then be represented by the operational expression.

$$e_{dn2}(t)=\left(\frac{p\cdot\gamma}{p+\gamma}\right)\frac{1}{h}\cdot\left(\frac{\sigma_0}{p+\sigma_0}\right)[\pi Q(1-1_\tau)]\qquad(15a)$$

$$\cong\frac{\gamma}{h}\left(\frac{\sigma_0}{p+\sigma_0}\right)[\pi Q(1-1_\tau)].\qquad(15b)$$

Performing the indicated operation we find then that the resulting pulse in the audio circuit, after having been modified by the frequency response of the audio circuit, has the following approximate form[6]:

$$e_{dn2}(t)=\left(\frac{\gamma\pi QE_{dm}}{\delta_0}\right)[1-\exp(-\sigma t)]\text{ for }0\leqq t\leqq\tau/\delta_0$$

$$=\left(\frac{\gamma\pi QE_{dm}}{\delta_0}\right)[\exp\sigma(\tau-t)-\exp(-\sigma t)]\qquad\text{for }t>\tau/\delta_0.\quad(16)$$

Bearing in mind that τ is given in the earlier part of the paper in units of time of $1/\delta_0$ instead of seconds, the peak value of $e_{dn2}(t)$ may be calculated, giving

$$[e_{dn2}(t)]_{\max}=\frac{\gamma\pi Q\sigma_0\tau}{\delta_0^2}\cdot E_{dm}.\qquad(17)$$

Hence, the click noise after transition through the audio amplifier has the following form. It builds up quickly at a rate determined by the top frequency response of the audio amplifier to a maximum value given by (17) and then decays exponentially at the same rate.

The amplitude of such a click pulse is considerably less than that of a pop noise, first because of the smaller value of Q, and second because of attenuation and integration of the pulse in the audio amplifier. For example,

[6] There is one unimportant exception to this which obtains when the noise and program carrier vectors pass through phase coincidence during the capture time, in which case the resultant noise form is roughly of the form of the time derivative of $e_{dn_2}(t)$.

for the case given above of 75-kilocycle maximum deviation, 100-microsecond time constant, and if in addition the audio amplifier has a high-frequency cutoff of 15 kilocycles, the maximum value of e_{dn3} for $\tau=3$ would correspond to about 4 per cent modulation instead of 13 per cent for the pop noise, and most of the clicks would be considerably less than 4 per cent. Further, the duration of the clicks are considerably less, since they decay at a rate determined by the high-frequency cutoff of the audio amplifier instead of by the considerably longer time constant of the de-emphasis circuit.

In all of the above cases we have considered large noise transients. If, however, the noise transient is weaker than the program signal, no capture obtains and the resultant noise will be similar to but weaker than a click noise. This condition has not been delineated here because the resulting noise is negligible for most practical receivers.

Conclusions

From the above analysis, quite a few interesting conclusions about impulse noise in frequency-modulation receivers may be drawn. First, any given noise impulse may produce either one of two kinds of noise: a faint click, or a louder pop. The click type is characterized by being high-pitched in tone quality and variable in amplitude, but always faint. The wider the audio amplifier passband, the louder and higher pitched the click. Conversely, the louder pop is of lower pitch, being determined by the time constant of the de-emphasis circuit, and is more or less independent of the frequency response of the audio amplifier.

Second, the amplitude of a click is largely, and of a pop completely, independent of the amplitude of the original noise impulse, and in an ideal system is uniquely determined by the constants of the receiver.

Third, whether a noise impulse will produce a click or a pop is largely a matter of random chance as far as the noise impulse is concerned, although due to the slight increase in capture time with increasing signal strength the chances of producing a pop are somewhat greater with stronger signals.

Fourth, the chances of a noise impulse producing a pop instead of a click increase linearly with the degree of instantaneous modulation and with mistuning. In fact, the whole analysis emphasizes the need for accurate tuning in frequency-modulation receivers.

Both the existence and wave shape as well as the probabilities of occurrence of clicks and pops have been fully verified by experiments conducted for us by C. T. McCoy of the Philco Research Laboratories. It is hoped that this experimental work will be the subject of a later paper.

Finally, one practical example illustrating the above conclusions should be mentioned. An electric drill or razor produces a series of noise impulses which vary considerably in amplitude from impulse to impulse. However, if one listens to a frequency-modulation program on a properly tuned high-quality receiver subject to such interference, one will hear only a few interrupting noises which will sound like and are, in fact, pops of the type described above. All of these pops in the audio signal will be found to be of the same amplitude. Further, if the modulation of program signal is removed, leaving only an unmodulated carrier, and if the receiver is then mistuned, it will be observed that, with mistuning, the number of interruptions or noises increases about linearly but again the volume of each individual noise is the same. Further, one will then hear the anticipated background of sizzling clicks, much weaker than the pops but nevertheless there. This is perhaps the most striking verification of the above theory.

Acknowledgment

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Appendix I

For the purposes of this paper we are concerned, not with the specific form of frequency-response or phase characteristic of our receiver, but only with the broad form of frequency selection. Hence, for analytical purposes we shall assume a tuner comprising a number of identical stages each consisting of a parallel-tuned circuit of inductance L, capacitance C, and conductance G, driven by a one-way current transducer having the characteristic $i_k=g_m\cdot e_{k-1}$. The operational form for the transfer impedance of each stage then is

$$\frac{1}{Y_{(p)}}=\frac{g_m}{C}\cdot\frac{p}{(p+\alpha)^2+\omega_0^2} \tag{18}$$

where

$$\alpha=\frac{1}{2}\frac{G}{C}$$

$$\omega_1^2=\frac{1}{LC}$$

$$\omega_0^2=\omega_1^2-\alpha^2$$

and p is the usual Heaviside operator. Then the output voltage e_n across the nth circuit is given by

$$e_n(t)=\left[\frac{g_m}{C}\right]^n\cdot\left[\frac{p}{(p+\alpha)^2+\omega_0^2}\right]^n e_0(t) \tag{19}$$

where e_0g_m is the current driving the first stage.

For a continuous-wave signal, the maximum response will obtain for the frequency ω_1 which is also the mid-band frequency. Hence, if the continuous-wave signal frequency is ω we let $\omega=\omega_1+v$, where v is the departure of the continuous-wave signal from the center frequency. If the continuous-wave signal is frequency-modulated at a relatively slow rate and stays within the channel, then v can be considered to be the deviation. Then the gain and frequency response of a single stage is given by

$$\frac{e_1}{e_0} = -\frac{g_m}{G}\left[\frac{+1}{1+j\frac{v}{\alpha}}\right]. \tag{20}$$

It will be noted that the frequencies at which the response is 3 decibels down are $v=\pm\alpha$. We will use the 3-decibel point to define the edges of the passband of the tuner, and we will define δ_0 (in terms of angular frequency) as the half-bandwidth of the system. Then, if the system consists of one stage, $\alpha=\delta_0$ and the conductance of the stage would be adjusted accordingly.

For n stages, the gain and frequency response is given by

$$\frac{e_n}{e_0} = \left[\frac{g_m}{G}\right]^n\cdot\left[\frac{+1}{1+j\frac{v}{\alpha}}\right]^n. \tag{21}$$

The damping α for a passband δ_0 at the 3-decibel point is determined from

$$\left\|\frac{+1}{1+j\frac{\delta_0}{\alpha}}\right\|^n = \frac{1}{\sqrt{2}}$$

or

$$\frac{\delta_0}{\alpha} = \left\{2^{1/n}-1\right\}^{1/2}.$$

Hence, for a tuner of n stages and passband δ_0 and center frequency ω_1, the gain is given by

$$\frac{e_n}{e_0} = \left[\frac{g_m}{G}\right]^n \quad \text{at} \quad \omega=\omega_1 \tag{22a}$$

and the natural decrement of each stage by

$$\alpha = \frac{\delta_0}{\left\{2^{1/n}-1\right\}^{1/2}}. \tag{22b}$$

impulse. Then for a single stage

$$N_1(t) = \left[\frac{g_m}{C}\right]\left[\frac{p}{(p+\alpha)^2+\omega_0^2}\right]Dp1 \tag{23}$$

where $N_1(t)$ is the resulting form. Hence,

$$N_1(t) = \frac{D\cdot g_m}{C}\cdot\exp(-\alpha t)\cdot\cos\omega_0 t \tag{24a}$$

which has a peak value

$$N_{1\,\max} = \frac{D\cdot g_m}{C}.$$

The ratio of noise peak to continuous-wave signal for a single stage is

$$\frac{N_{1\,\max}}{E_1} = \frac{D\cdot g_m}{E_0\cdot C}\cdot\frac{G}{g_m} = \frac{2D\cdot\alpha}{E_0} \tag{24b}$$

where E_0 is the peak value of the desired input signal envelope. For n stages

$$N_n(t) = \left[\frac{g_m}{C}\right]^n\left[\frac{p}{(p+\alpha)^2+\omega_0^2}\right]^n D\cdot p1. \tag{25a}$$

This expression may be most easily evaluated by use of the contour integral form

$$N_n(t) = \frac{D}{2\pi j}\left[\frac{g_m}{C}\right]^n\cdot\int_{-j\infty}^{j\infty}\frac{z^n\cdot\exp t\cdot z}{[(z+\alpha)^2+\omega_0^2]^n}\cdot dz \tag{25b}$$

The above integral has two poles, each of order n, at

$$\left.\begin{aligned}\lambda_1 &= -\alpha+j\omega_0\\ \lambda_2 &= -\alpha-j\omega_0.\end{aligned}\right\} \tag{25c}$$

The contour may be closed for all positive values of t by a semicircle of infinite radius from $+j\infty$ to $-j\infty$ enclosing the poles. The value of the integral is hence $2\pi j$ times the sum of the residues at λ_1 and λ_2. These may be found in the usual way by expanding the integral in a Taylor's series around the poles. Hence,

$$N_n(t) = D\cdot\left[\frac{g_m}{C}\right]^n\cdot\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[\left(\frac{z}{z-\lambda_2}\right)^n\exp tz\right]_{z\to\lambda_1}+\frac{d^{n-1}}{dz^{n-1}}\left[\left(\frac{z}{z-\lambda_1}\right)^n\exp tz\right]\right\}_{z\to\lambda_2}. \tag{25d}$$

The behavior of such a system when shock-excited is as follows: The initial exciting force is taken as $e_0=D\cdot p1$ where D has the dimensions of the product of voltage envelope and time. This function has an equal distribution of energy with frequency. An actual impulse noise of duration short as compared with $1/\delta_0$ will have an energy distribution with frequency which will be a maximum at its nominal carrier and fall off with frequency at a rate determined by its shape. However, over the acceptance band of the filter the energy distribution is substantially uniform. Hence, the spike function $Dp1$ will give the proper transient wave shape, but quantitative values of D must be used with caution and regard to the actual energy distribution of the noise

For convenience we write the derivatives, respectively, as

$$\begin{aligned}\psi_1(\lambda_1,\lambda_2) &= \frac{d^{n-1}}{dz^{n-1}}\left[\left(\frac{z}{z-\lambda_2}\right)^n\exp tz\right]_{z\to\lambda_1}\\ \psi_2(\lambda_2,\lambda_1) &= \frac{d^{n-1}}{dz^{n-1}}\left[\left(\frac{z}{z-\lambda_1}\right)^n\exp tz\right]_{z\to\lambda_2}.\end{aligned} \tag{25e}$$

Since λ_1 and λ_2 are conjugate functions, ψ_1 and ψ_2 are likewise conjugate functions. Hence, we need treat only ψ_1. First removing the exponent from under the derivative

$$\psi_1(\lambda_1\lambda_2) = \exp\lambda_1 t\left[t+\frac{d}{dz}\right]^{n-1}\cdot\left[\frac{z}{z-\lambda_2}\right]^n_{z\to\lambda_1} \tag{25f}$$

where the expression in the first bracket is to be treated as an operator operating $n-1$ times on the succeeding terms. In this particular case, it is legitimate to expand the operator in a binomial series, hence

$$\psi_1(\lambda_1\lambda_2) = \exp \lambda_1 t \cdot \sum_{k=0}^{k=n-1} \left(\frac{k}{n-1}\right) \cdot t^{n-1-k} \cdot \frac{d^k}{dz^k}\left[\frac{z}{z-\lambda_2}\right]^n_{z\to\lambda_1}. \quad (25g)$$

The principle term of this series is the first for which $k=0$. This term is as follows:

$$\exp \lambda_1 t \cdot t^{n-1} \cdot \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)^n = \frac{1}{2}\left(\frac{t}{2}\right)^{n-1} \cdot \left\{1+\left(\frac{\alpha}{\omega_0}\right)^2\right\}^{n/2} \exp\left[(-\alpha+j\omega_0)t+jn\theta\right]\Big\} \quad (25h)$$

where $\theta=\tan^{-1}(d/\omega_0)$. Noting that $\alpha/\omega_0 \ll 1$, neglecting the phase angle, and adding the conjugate term in ψ, we have approximately

$$N_n(t) \cong \frac{D\cdot g_m}{C}\cdot\left[\frac{g_m}{G}\right]^{n-1}\cdot\frac{(\alpha t)^{n-1}}{(n-1)!}\cdot \exp - \alpha t \cdot \cos \omega_0 t. \quad (25i)$$

The higher terms ($k>0$) if added to (25i) will give terms of the following form and very approximate value,

$$\left\{\frac{D\cdot g_m}{C}\cdot\left[\frac{g_m}{G}\right]^{n-1}\cdot\right\}\cdot\left(\frac{k}{N}\right)\left(\frac{\alpha}{2\omega_0}\right)^k\frac{(\alpha t)^{n-1-k}}{(n-1-k)!} \cdot \exp - \alpha t \cos\cdot\left(\omega_0 t+\frac{k\pi}{2}\right). \quad (25j)$$

Expression (25i) represents an oscillation which has an envelope given by

$${}_eN_n(t) = \frac{D\cdot g_m}{C}\left[\frac{g_m}{G}\right]^{n-1}\cdot\frac{(\alpha t)^{n-1}}{(n-1)!}\exp(-\alpha t). \quad (26a)$$

Equating the time derivative to zero we find the maximum value of the envelope obtains for $\alpha t=n-1$. Hence,

$${}_eN_{n\max} = \frac{D\cdot g_m}{C}\left[\frac{g_m}{G}\right]^{n-1}\cdot\frac{(n-1)^{n-1}\exp(1-n)}{(n-1)!} \quad (26b)$$

and the form of the signal in terms of its peak value is

$$y_n(t) = \frac{{}_eN_n(t)}{{}_eN_{n\max}} = \left(\frac{\alpha t}{n-1}\right)^{n-1}\cdot\frac{\exp(-\alpha t)}{\exp(1-n)}. \quad (26c)$$

The higher terms given in (25j) above come to their maximum earlier at $\alpha t=n-1-k$. These peaks, however, are smaller in amplitude than the principal term for all important values of t. They are displaced in phase by approximately ninety degrees per term. The net effect is to introduce a small amount of phase or frequency modulation which, however, finally settle down to the natural frequency ω_0. For our purposes, this modulation plus such minor variations in the envelope as accompany it may be neglected.

From (22a) and (26a) we can calculate the noise-peak to output-signal ratio, which is given by

$$\frac{{}_eN_{n\max}}{E_n} = \frac{2\cdot D\cdot\alpha}{E_0}\cdot\frac{(n-1)^{n-1}\exp(1-n)}{(n-1)!}. \quad (27a)$$

Including the variation in bandwidth of individual stages with the number of stages as given in (22b), this becomes

$$\frac{{}_eN_{n\max}}{E_n} = \frac{2\cdot D\cdot\delta_0}{E_0}\cdot\frac{(n-1)^{n-1}\exp(1-n)}{\{2^{1/n}-1\}^{1/2}(n-1)!} = S. \quad (27b)$$

We now wish to determine the time interval during which the noise impulse exceeds the desired signal for various signal-to-noise ratios. Let τ/δ_0 be that time interval and $y_n(t)$ the ratio of the noise envelope at time t to the maximum noise envelope peak. We can assume the signal envelope to be of constant amplitude. Hence the ratio $S\cdot y_n(t)$ of noise envelope to signal as a function of time from (26c) is

$$S\cdot y_n(t) = S\cdot\left(\frac{\alpha t}{n-1}\right)^{n-1}\exp[n-1-\alpha t] \quad (28)$$

where S is the peak-noise to signal ratio.

This expression is zero for $t=0_1\infty$ and equals S for $\alpha t=n-1$. The time interval τ/δ_0 is given by $\tau\delta_0$ x_2-x_1 where x_2 and x_1 are the roots of the transcendental equation

$$S\left(\frac{\alpha t}{n-1}\right)^{n-1}\exp[n-1-\alpha t] = 1. \quad (29a)$$

To solve this equation we substitute $x=\alpha t-n+1$, then, taking the log of each side and rearranging the terms,

$$(n-1)\ln\left[\frac{x}{n-1}+1\right]-x = -\ln S. \quad (29b)$$

Using the first few terms of the series for the logarithm we have

$$x-\frac{x^2}{2(n-1)}-x = -\ln S \quad (30a)$$

or

$$x = \pm\sqrt{2(n-1)\ln S}. \quad (30b)$$

Hence,

$$\frac{\tau}{\delta} = \frac{2}{\alpha}\sqrt{2(n-1)\ln S} \quad (31a)$$

$$\tau = 2\{2^{1/n}-1\}^{1/2}\{2(n-1)\ln S\}^{1/2}. \quad (31b)$$

It will also be noted that the transcendental equation may easily be solved graphically by plotting on semilog paper.