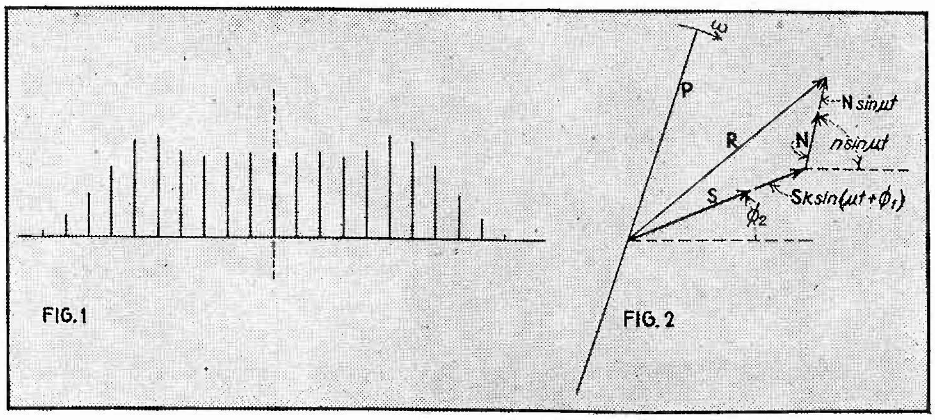

Fig. 1—Side-band content of assumed noise voltage. Fig. 2—Noise voltage superimposed on amplitude modulated signal

Noise in Frequency Modulation

The first published mathematical demonstration of the validity of the noise-suppression effect in wide-band frequency modulation, by one of the outstanding authorities on modulation theory, which shows the necessity of a wide band and of proper "limiter" action

EDITOR'S NOTE: The mathematical operations in Mr. Roder's paper are not so involved as they appear at first glance; for the most part only trigonometric operations and rotating vector-diagrams are employed. The interested reader is invited, therefore, to "dig in". As a preliminary, however, the following notes may be helpful:

Noise voltages encountered in radio frequency circuits are modulated both in amplitude and in frequency. When added to a conventional amplitude-modulated signal, such double-modulated noise produces disturbances in the *amplitude* of the modulated signal, and these disturbances appear in the output of the detector. To overcome the effect of these disturbances, the amplitude of the desired signal can be increased, relative to the noise, by increasing the power and/or the percentage modulation of the transmitter. The practical limit to this procedure is reached when the power can no longer be economically increased.

When the double-modulated noise is applied to a frequency-modulated signal of constant amplitude, however, the noise produces two effects. First it introduces an amplitude disturbance to the desired signal. By the use of an amplitude-limiter in the i-f circuit of the receiver, these amplitude disturbances are removed and hence do not affect the audio output of the converter-detector, which is responsive both to frequency and amplitude variations. In the second place the noise, because of its frequency-modulated component, produces *disturbances in the frequency-swing* of the desired signal. These latter disturbances do affect the converter-detector and are present in the audio output. To overcome the effect of these disturbances, the frequency-swing of the desired signal must be increased with respect to the frequency-swing of the noise. The percentage modulation of the signal is thereby increased relative to the noise, and this can be done without increasing the power of the transmitter. Thus, the wider the frequency swing of the desired signal, the higher the signal-to-noise ratio. The degree of improvement depends on the amplitude of the noise relative to that of the signal. When the noise amplitude is not greater than one-half the signal amplitude, however, and when the frequency swing of the desired signal is 100,000 cycles compared with 7000 cycles for the noise, the interference factor is not greater than one percent throughout the audio signal range from 100 to 5000 cycles. The same noise, applied to an amplitude-modulated signal under the same conditions (noise amplitude one-half signal amplitude), would produce an interference factor of the order of 50 percent or higher.

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THE proposal to use frequency modulation in place of amplitude modulation was made early in radio development, but the hope of obtaining thereby greatly reduced band-widths was shattered in 1922 when Carson[1] published his fundamental analysis. Despite his adverse criticism, much experimental research was done in the following years. In 1936, Major E. H. Armstrong announced[2] and demonstrated the phenomenon of noise suppression with wide-band frequency modulation. This effect was made possible by the use of the ultra-high frequencies, which permitted the use of a band-width about 10 times greater than customary in amplitude modulation. The explanation of the noise-suppression effect has not, to the author's knowledge, been given a simple and satisfactory treatment. In view of the contrast between the predictions of earlier theories and the recent experimental results, it is desirable to extend the theory to the field of ultra-high frequencies and to greatly extended side-band coverage, as has been done in the practical attack on the problem.

This extension of theory is presented in the following paper. While the treatment refers to a simplified case, it will permit us to draw some interesting conclusions in regard to the system. The plan is as follows: We set up a convenient type of noise signal, which is both amplitude and frequency modulated, and we apply this noise signal first to a standard amplitude-modulated signal and second to a wide-band frequency modulated signal. We can then compare the effect of the noise on the two types of modulation.

The Form of the "Noise Signal"

The "noise signal" on which the analysis is based has the following form: Its basic carrier frequency is $\omega/2\pi$. It is 100 per cent amplitude-modulated by a modulation frequency $\mu/2\pi$. It is frequency modulated by the same modulation frequency $\mu/2\pi$ and the maximum frequency shift involved in the frequency modulation is $\Delta\omega/2\pi$. Its mathematical form is:

$$e_n = N\,(1 + \sin \mu t) \sin\left(\omega t + \frac{\Delta\omega}{\mu} \sin \mu t\right) \quad (1)$$

The ratio $\frac{\Delta\omega}{\mu}$, which we shall give the symbol n, is the maximum number of radians phase shift involved in the frequency modulation. As a convenient numerical example we can take a modulation frequency of 1000 cycles, and a maximum frequency shift of $\pm$ 7000 cycles, so that $n = 7$ radians or 402°. With this numerical value, Eq. (1) can be analyzed[3] to determine its side-band frequency content. It is found that there are about 12 sidebands on each side of the carrier, the spacing between each being 1000 cycles. The strongest sidebands are between 2000 and 8000 cycles, and all the relative phases and amplitudes of adjoining sidebands are different. Because this signal covers a wide band (24,000 cycles total) and has many sideband components, all of different amplitudes and phases, it is a good representation of a "noise voltage." Its side-band content is represented graphically in Fig. 1.

"Noise" Superimposed on an Amplitude-Modulated Signal

Having set up a typical noise voltage, e_n, we apply it first to an amplitude modulated signal e_a, whose carrier frequency is $\omega/2\pi$, and whose modulation frequency $\mu/2\pi$, as in the case of the noise. The percentage modulation is $K \times 100$ per cent, and the phase angles of the modulation and carrier voltages are ϕ_1 and ϕ_2 respectively. The form of this amplitude-modulated signal is:

$$e_a = S\,(1 + K \sin(\mu t + \phi_1)) \sin(\omega t + \phi_2) \quad (2)$$

The resultant voltage at the receiver, due to noise plus desired modulated signal, is found by adding Eqs. (1) and (2). If this resulting signal is detected by a perfectly linear rectifier, and if the electrical network preceding the detector is wide enough to pass all sidebands, then the output of the rectifier will be a perfect reproduction of the envelope of Eq. (1) plus Eq. (2). The form of this envelope is most conveniently found by graphical means, as shown in Fig. 2.

In Fig. 2, the projection axis P is rotating clockwise at an angular velocity of ω radians per second. The desired signal is given by the two vectors S and $S\,K \sin(\mu t + \phi_1)$. These two parts are at a fixed phase angle ϕ_2. The vector expands and contracts $\mu/2\pi$ times per second between the limits $S - SK$ and $S + SK$.

The noise voltage is likewise in two parts, N and $N \sin \mu t$, but since it is both frequency and amplitude modulated, is performing a pendulum motion while contracting and expanding. The instantaneous phase angle, which measures the pendulum motion, is $n \sin \mu t$ radians, the phase changing from $+n$ to $-n$ radians $\mu/2\pi$ times per second. The length of the noise vector expands to $2N$ and contracts to zero, $\mu/2\pi$ times per second. During one audio cycle, the end of the noise vector $N\,(1 + \sin \mu t$ describes a spiral. The vector addition of both the noise and desired signals yields the resultant signal R, which may be plotted versus time over one audio cycle, thus giving the desired envelope of Eqs. (1) + (2). Figure 3 shows the resulting envelopes for various values of the angles ϕ_1 and ϕ_2, under the conditions that $N = S/2$ (noise amplitude one half of signal amplitude) $K = 1$ (100 per cent modulation of signal carrier) and $n = 7$ radians (as above). The heavy lines in Fig. 3 give the resultant envelope, the dotted lines the undisturbed signal envelope. It is evident that the harmonic content of the resultant envelope is excessive, that is, that the desired signal is badly damaged if not rendered completely useless by the superposition of the noise voltage.

Noise Superimposed on Wide-band Frequency-modulated Signal

We now apply the noise voltage e_n to a wide-band frequency-modulated signal e_f, as follows: The wide-band

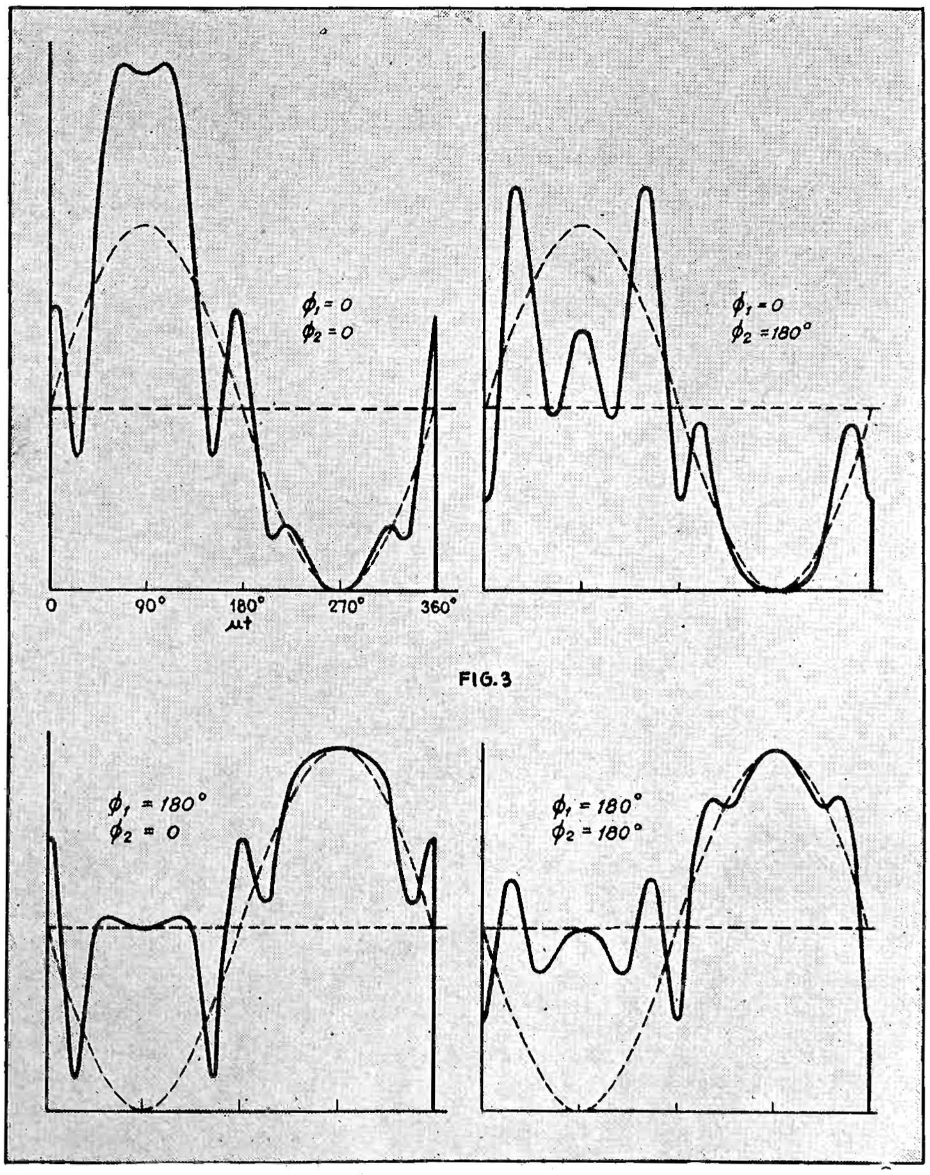


Fig. 3—Envelopes of amplitude modulated signal with (solid) and without (dotted) noise superimposed

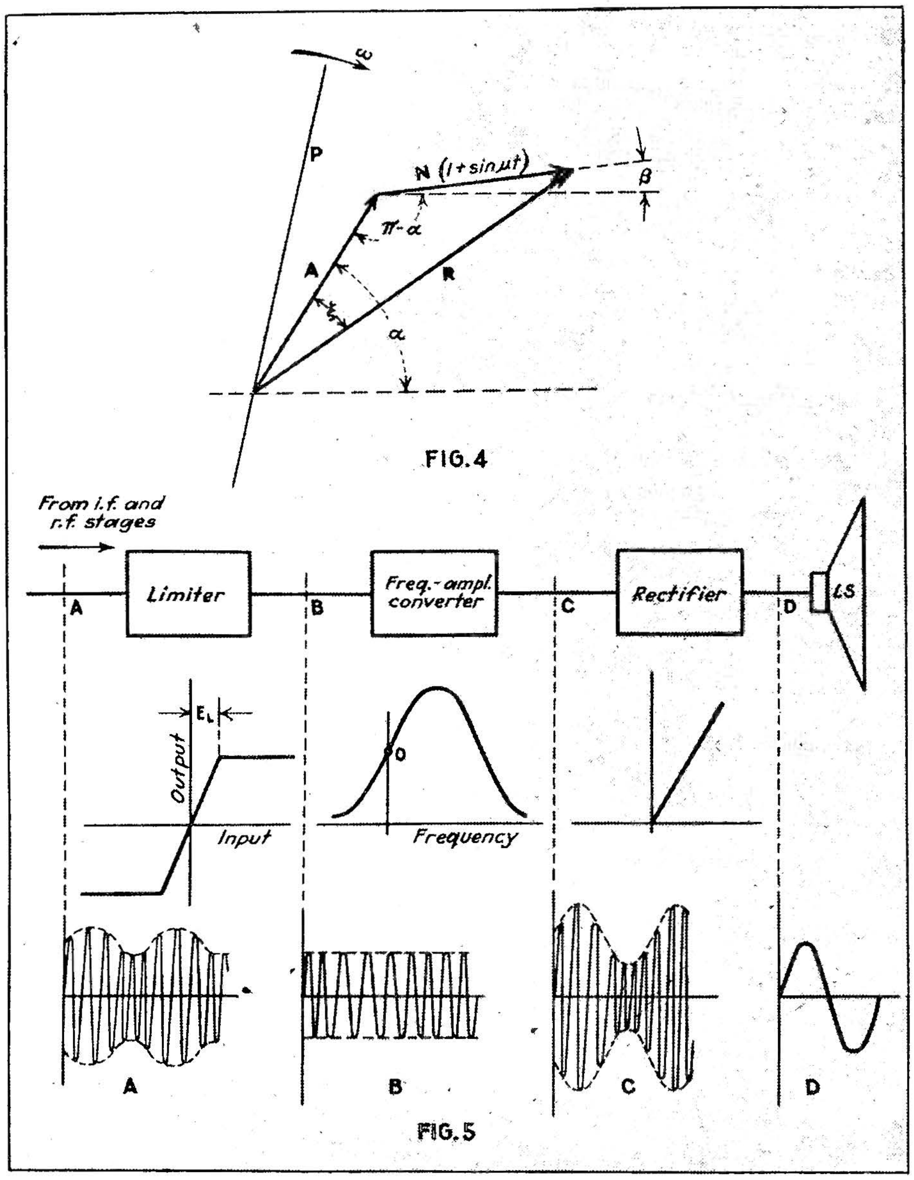


Fig. 4—Noise superimposed on frequency-modulated signal
Fig. 5—Elements of the frequency-modulated receiver

frequency-modulated signal has a constant carrier amplitude A, a central carrier frequency $\omega/2\pi$, a maximum frequency shift $\Delta\omega/\pi$, and a modulation frequency $\mu/2\pi$. The form of the frequency modulated signal is:

$$e_f = A \sin\left(\omega t + \frac{\Delta\omega}{\mu} \sin \mu t\right) \qquad (3)$$

The ratio $\frac{\Delta\omega}{\mu}$, which in this case is given the symbol m, is the maximum number of radians phase shift in the frequency-modulated signal. In the wide-band signal used by Major Armstrong, the frequency shift $\frac{\Delta\omega}{2\pi}$ is of the order of $\pm$ 100,000 cycles. For a modulation frequency of 1000 cycles, therefore, the value of m is 100 radians or 5729°.

We obtain the resultant voltage, as before, by adding the noise with the desired signal, that is, Eq. (1) plus Eq. (3), and determine its envelope by graphical construction, shown in Fig. 4. The projection vector P rotates ω radians per second. The signal vector is of constant amplitude A but its phase $\alpha = m \sin \mu t$ changes from $+ m$ to $- m$, $\mu/2\pi$ times per second. To this vector is added the noise vector, which is both amplitude- and frequency-modulated. The phase of the noise vector is $\beta = n \sin \mu t$. The noise and signal combine vectorially to give the resultant vector R, whose instantaneous phase is $\alpha - \xi$. The angle ξ between A and R is determined by the instantaneous phase and amplitude of the noise voltage. In evaluating the effect of the noise on the desired signal, the resultant voltage vector R may be compared with the desired signal A itself. In so doing, it is necessary to consider the manner in which the frequency-modulation receiver operates in converting the frequency-modulated signals into audio signals.

The Frequency-modulated Receiver

In Fig. 5 are shown the essential elements of the frequency-modulated receiver. The r-f and i-f stages, which simply amplify all sideband frequencies proportionately, are not shown. The first essential element in the receiver is the limiter which removes any amplitude variations from the signal. These amplitude variations are introduced not only by the addition of the noise signal which is partially amplitude-modulated, but to some extent also by frequency discrimination in preceding tuned circuits. The signal leaving the limiter, containing only frequency variations, enters the frequency-amplitude converter (essentially a tuned circuit with the maximum slope of its resonant curve tuned to the signal carrier) which converts the frequency variations directly into amplitude variations. These are then detected in a linear rectifier which produces the audio signal. It is assumed that all conversion and detection is perfectly linear throughout the range of sideband frequencies involved.

After the signal leaves the limiter we are interested in its instantaneous frequency, since this instantaneous frequency is proportional to the instantaneous amplitude at the out-

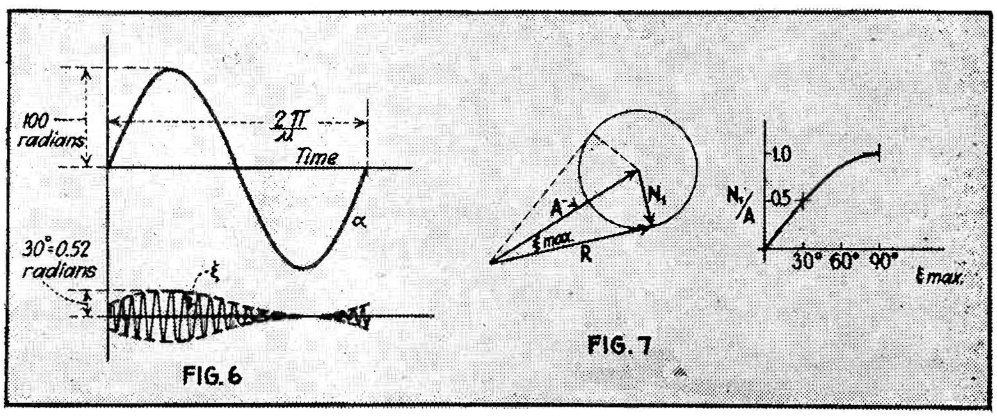


Fig. 6—Relative amplitudes of phase shift in signal and noise
Fig. 7—Maximum phase disturbance due to noise

put of the converter. The instantaneous angular frequency of any vector whose phase ϕ is variable is by definition $\omega = 2\pi f = d\phi/dt$, hence the frequency presented to the converter at any instant is:

$$f_{inst} = \frac{1}{2\pi} d(\alpha - \xi)/dt \qquad (4)$$

since $\alpha - \xi$ is the instantaneous phase of the resultant voltage R (see Fig. 4). To solve Eq. 4, ξ must be found in terms of the desired-signal and noise-voltage parameters. The vector diagram in Fig. 4 reveals that:

$$\frac{\sin\xi}{\sin(\alpha - \beta - \xi)} = \frac{N(1 + \sin\mu t)}{A}$$

To facilitate the limiting process, the maximum noise amplitude $2N$ should not be greater than one-half the signal amplitude A. In other words the maximum value which the ratio $\sin\xi/\sin(\alpha - \beta - \xi)$ can have is 1/2 and the maximum value of ξ is then 30°. We therefore make the approximations $\sin\xi = \xi$ and $\cos\xi = 1$, and write the expression for ξ:

$$\xi = \frac{k(1 + \sin\mu t)\sin(\alpha - \beta)}{1 + k(1 + \sin\mu t)\cos(\alpha - \beta)} \qquad (5)$$

where $k = \frac{N}{A}$

Equation (5) may be made more convenient to use by substituting a change of variables and expanding in a series, as follows: Let $y = 1 + \sin\mu t$ and $\gamma = \alpha - \beta = m\sin\mu t - n\sin\mu t = (m-n)\sin\mu t$. Then:

$$\xi = k\,y\sin\gamma \frac{1}{1 + k\,y\cos\gamma} \qquad (6)$$

This equation can be expanded in a converging power series, since $ky\cos\gamma$ is equal to or less than 1/2. The resulting series is given in Eq. (11) in the appendix. If as an approximation we neglect all terms containing k to the second or higher powers, this series expression for ξ becomes

$$\xi = 2\sqrt{\frac{1}{\pi(m-n)}}(\sin\mu t - \sin 3\mu t + \sin 5\mu t - ..)$$
$$(\cos(m-n) - \sin(m-n) \cdot ky) \qquad (7)$$

If we substitute this expression for ξ in Eq. (4), together with the expressions $\alpha = m\sin\mu t$, and $y = 1 + \sin\mu t$, and differentiate, we obtain the instantaneous frequency f_{inst} which is presented to the input of the frequency-amplitude converter, as follows:

$$f_{inst} = \frac{\mu}{2\pi}\left(m\cos\mu t - 2k\sqrt{\frac{1}{\pi(m-n)}}\right.$$
$$(\cos(m-n) - \sin(m-n))$$
$$(\cos\mu t + 2\sin 2\mu t - 3\cos 3\mu t - 4\sin 4\mu t + 5\cos 5\mu t + 6\sin 6\mu t - 7\cos 7\mu t - 8\sin 8\mu t + \ldots\ldots) \qquad (8)$$

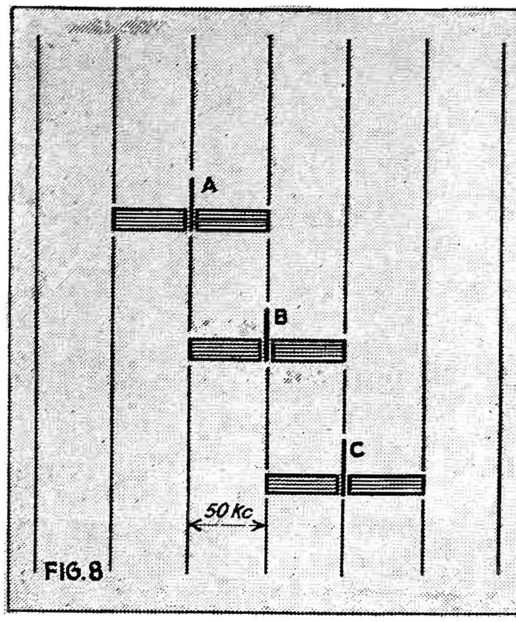
Since this instantaneous frequency is proportional to the rectifier's audio output, the term $m\cos\mu t$ represents the desired audio frequency, while all the other terms are spurious frequencies introduced by the presence of the noise voltage. It will be seen that these spurious frequencies are multiples of the original audio frequency $\mu/2\pi$ and that their amplitude increases in direct proportion to their frequency. Fortunately only the lower frequency components in this interference spectrum are of practical importance because the audio system of the receiver (and the ear) will not respond to frequencies much higher than ten thousand cycles.

We set up the interference factor (which is the ratio of the amplitudes of the interference to that of the desired signal) directly from Eq. (8), and obtain

$$\text{Interference factor} = \frac{2k}{m}\sqrt{\frac{1}{\pi(m-n)}}$$
$$(\cos(m-n) - \sin(m-n))\sqrt{1^2 + 2^2 + 3^2 + \ldots Z^2} \qquad (9)$$

Now, remembering that $n = \Delta\omega/\mu$ for the noise signal, and $m = \Delta\omega/\mu$ for the frequency modulated signal, we can evaluate the interference factor for various combinations of audio frequencies and maximum frequency shifts. The greatest value which $(\cos(m - n) - \sin(m - n))$ can have is $\sqrt{2}$. The numerical value of the second root in (9) is computed from $1^2 + 2^2 + 3^2 + \ldots Z^2 = \frac{1}{6} Z (Z + 1)(2Z + 1)$. Table 1 has been calculated for a value of $k = 1/2$ (noise amplitude half of signal amplitude), and on the assumption that all interference beat fre-

(*Continued on page 60*)



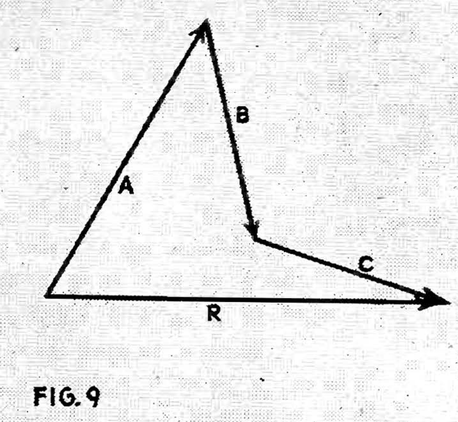


Fig. 8—Three over-lapping frequency-modulated stations assumed for purposes of interference analysis. Fig. 9—If R is larger than A, B, or C, no station is received

TABLE 1

Audio Freq.	Desired signal		Inter-ference signal			No. of inter-fer nce beats	Inter-ference factor	Notes
$\mu/2\pi$	$\Delta\omega$	m	$\Delta\omega$	n	m-n	Z	%	
100	100,000	1000	7000	70	930	100	0.760	Audio frequency variable. Per cent modulation of desired signal constant. Bandwidth of interfering signal constant.
500	100,000	200	7000	14	186	20	0.783	
1000 (A)	100,000	100	7000	7	93	10	0.810	
2000	100,000	50	7000	3.5	46.5	5	0.946	
5000	100,000	20	7000	1.4	18.6	2	1.03	
1000	100,000	100	10000	10	90	10	0.824	Audio frequency constant. Per cent modulation of desired signal constant. Interference signal bandwidth variable.
1000	100,000	100	20000	20	80	10	0.874	
1000	100,000	100	40000	40	60	10	1.01	
1000	100,000	100	80000	80	20	10	1.75	
1000	100,000	100	15,000	15	85	10	0.848	Audio frequency constant. Per cent modulation of desired signal variable. Interference signal bandwidth constant.
1000	50,000	50	15,000	15	35	10	3.65	
1000	20,000	20	15,000	15	5*	10	5.07*	
1000	10,000	10	15,000	15	—5*	10	10.14*	

*Formula (8) is correct only for large values of m-n. The above figures for m-n = 5 have been computed from a more accurate formula.

FREQUENCY MODULATION

(Continued from page 25)

quencies higher than 10,000 cps. are elimniated by the audio system (i.e. the value of Z in Eq. (9) does not go higher than $10{,}000/(\mu/2\pi)$. The table shows that the values of the interference factor are very small indeed, (below 1 per cent) particularly in case m is large compared with n. We may compare one case (marked (A) in the Table) directly with Fig. 3. The distortion in Fig. 3, in which the noise is added to an amplitude-modulated 1000-cycle signal, is excessive. When the same noise is added to a frequency-modulated 1000-cycle signal having a frequency shift of 100,000 cps., the interference factor is only 0.8 per cent. It will be noticed, however, that when the maximum frequency shift in the desired signal is not so great, the interference factor increases, being about ten per cent when the frequency shifts of noise and signal are approximately the same.

That the suppression of noise interference in frequency modulation thus depends on the use of a signal modulated over a wide frequency band can also be shown by considering equation (4), which states that the instantaneous frequency (proportional to audio output) is proportional to the derivative of $\alpha - \xi$. The phase shift of the desired signal $\alpha = m \sin \mu t$, can be made to have any desired maximum value; in the example a maximum value of 100 radians has been assumed. The angle ξ which is the phase disturbance produced by the noise, cannot have a maximum value higher than $30° = 0.52$ radians, if the noise amplitude is kept lower than 1/2 the signal amplitude. The disturbing phase shift is thus never more than about 1/200 of the legitimate phase shift, as shown in Fig. 6, where α and ξ are plotted against time. This small disturbance cannot produce much interference. The secret of interference suppression thus depends on making α, the phase shift of the legitimate signal, large compared with ξ, the phase disturbance produced by the noise. If we decide that α should be ten times as great as ξ, which is 0.52 radians if the noise amplitude is not more than 1/2 the signal amplitude, then α must be $0.52 \times 10 = 5.2$ radians. Thus, if the highest audio frequency to be transmitted is 10,000 cycles, then the maximum frequency shift required is $\pm$ 52,000 cycles.

Even when the maximum amplitude of the interfering noise is equal to that of the desired signal, the maximum value of ξ is comparatively small. This follows from the diagram in Fig. 7 which shows that sin $\xi_{max} = N_1/A$ where N_1 is the maximum noise amplitude and A the signal amplitude. For $N_1/A = 1$, therefore, $\xi_{max} = 90° = 1.57$ radians. If the desired signal phase shift is made 20 or more radians in this case, a very effective reduction of interference is obtainable.

The only additional requirement is that the difference $A - N_1$ must be large enough to permit operation of the limiter, i.e. it must be larger than E_L in Fig. 5. So long as this is true, the noise may be almost any kind of a signal. It may be wide-band or narrow-band, amplitude or frequency modulated, but its interference effect will always be small. *Thus to eliminate noise by frequency modulation, it is necessary to combine wide-band-frequency modulation with an effective limiting device at the receiver.*

The Effect of Interference Between Two or More Desired Signals

If two frequency-modulated transmitters are synchronized (carrier frequencies exactly the same) and are transmitting the same program simultaneously, the two signals will arrive at the receiver together. The question is what effect the two signals will have upon one another. By referring to Fig. 4, we see that $\alpha - \beta$ is constant, since both the radio and audio frequencies of the two signals are the same. Hence ξ is a constant also, and $d\xi/dt$ is zero. The two signals therefore do not produce any mutual interference, except in localities where they are of precisely the same amplitude and opposite phase, in which case they would cancel each other out. Thus in frequency modulation two synchronized signals will not interfere with one another so long as one signal is stronger than the other. This must not lead us to believe that transmitters may be set up, separated by 10 or 20 kc. in frequency, and operate simultaneously with a bandwidth of ± 100,000 cycles without interfering.

To show the fallacy in this case, assume three stations A, B and C each capable of 100 per cent modulation, producing a total bandwidth in each case of 200,000 cycles with carriers separated by 50,000 cycles, as shown in Fig. 8. If A operates 50 per cent modulated, the bandwidth is then 100,000 cycles. Now if station B comes on with a signal strength only 70 per cent of that of station A, station B cannot be tuned in because station A is stronger. This will occur even with the 50 kc. separation between carriers, since the receiver acceptance band must be 200,000 kc. to accept 100 per cent modulation. Now if station C joins in on the air, with a signal only 60 per cent as strong as A, in this case no station at all is received, because the sum of any two signal components is larger than the other, as shown in Fig. 9. To eliminate this effect, the carriers of the stations must be separated by a frequency interval equal to the pass-band of the receiver, namely 200,000 cycles in this case, which is the same requirement that exists in simple amplitude modulation. To bring in a week signal adjacent to a strong one requires a certain amount of selectivity in the receiving set. The effect of this selectivity on reception is an interesting problem which yet remains to be investigated.

Appendix: Derivation of Series for ξ

If we substitute in Eq. (5) $y = 1 + \sin \mu t$, $y = \alpha - \beta = (m - n) \sin \mu t$ we obtain for ξ:

$$\xi = ky \sin \gamma \frac{1}{1 + ky \cos \gamma} \qquad (6)$$

$$\xi = ky \sin \gamma \, (1 - ky \cos \gamma + (ky \cos \gamma)^2 - (ky \cos \gamma)^3 + \ldots$$

$$\xi = ky \sin \gamma - k^2y^2 \, 1/2 \sin 2\gamma + k^3y^3 \, 1/4 (\sin 3\gamma + \sin \gamma) - k^4y^4 \, 1/8 (\sin 4\gamma + 2 \sin 2\gamma) + k^5y^5 \, 1/16 (\sin 5\gamma + 3 \sin 3\gamma + 2 \sin \gamma) - \ldots .$$

By rearranging

$$\xi = + ky \sin \gamma \, (1 + \tfrac{1}{4} k^2y^2 + 2/16 \, k^4y^4 + 5/64 \, k^6y^6 + \ldots) - \tfrac{1}{2} (ky)^2 \sin 2\gamma \, (1 + 2/4 \, k^2y^2 + 5/16 \, k^4y^4 + 14/64 \, k^6y^6 + \ldots) + \tfrac{1}{4} (ky)^3 \sin 3\gamma \, (1 + \tfrac{3}{4} k^2y^2 + 9/16 \, k^4y^4 + \ldots\ldots\ldots) - 1/8 \, (ky)^4 \sin 4\gamma \, (1 + 4/4 \, k^2y^2 + 14/16 \, k^4y^4 + \ldots) + 1/16 \, (ky)^5 \sin 5\gamma \, (1 + 5/4 \, k^2y^2 + \ldots) \qquad (10)$$

Using a well known expansion due to Neumann, the terms $\sin \gamma = \sin ((m - n) \sin \mu t)$ can be developed into the following series:

$$\sin \gamma = 2 J_1 (m-n) \sin \mu t + 2 J_3 (m-n) \sin 3 \mu t + 2 J_5 (m-n) \sin 5 \mu t + \ldots .$$

The terms $J_p (m - n)$ are Bessel's functions of the first kind, order p, argument $(m - n)$. Now, the term $(m - n)$ is large for instance, $m = 100$ and $n = 7$) therefore the coefficients $J_p (m - n)$ can be expressed by trigonometric functions as is shown in the theory of Bessel's functions:

$$J_1 (m-n) = + \sqrt{\frac{1}{\pi (m-n)}} (\cos (m-n) - \sin (m-n))$$

$$J_3 (m-n) = - \sqrt{\frac{1}{\pi (m-n)}} (\cos (m-n) - \sin (m-n))$$

$$J_5 (m-n) = \sqrt{\frac{1}{\pi (m-n)}} (\cos (m-n) - \sin (m-n))$$

and so on. We then have

$$\sin \gamma = 2 \sqrt{\frac{1}{(m-n) \pi}} (\cos (m-n) - \sin (m-n)) (\sin \mu t - \sin 3 \mu t + \sin 5 \mu t \ldots)$$

$$\sin 2\gamma = 2 \sqrt{\frac{1}{2 (m-n) \pi}} (\cos 2 (m-n) - \sin 2 (m-n)) (\sin \mu t - \sin 3 \mu t + \sin 5 \mu t)$$

$$\sin 3\gamma = 2 \sqrt{\frac{1}{3 (m-n) \pi}} \cos 3 (m-n) - \sin 3 (m-n)) (\sin \mu t - \sin 3 \mu t + \sin 5 \mu t)$$

By substitution of these three expressions into (10) and by re-arranging, one obtains the following:

$$\xi = 2 \sqrt{\frac{1}{\pi (m-n)}} (\sin \mu t - \sin 3 \mu t + \sin 5 \mu t - \sin 7 \mu t + \ldots) \Big[+ \frac{1}{\sqrt{1}} (\cos (m-n) - \sin (m-n)) \, ky \, (1 + \tfrac{1}{4} (ky)^2 + 2/16 \, (ky)^4 + 5/64 \, (ky)^6 - \frac{1}{\sqrt{2}} (\cos 2 (m-n) - \sin 2 (m-n)) \, \tfrac{1}{2} k^2y^2 (1 + 2/4 \, k^2y^2 + 5/16 \, k^4y^4 + 14/64 \, k^6y^6 + \frac{1}{\sqrt{3}} (\cos 3 (m-n) - \sin 3 (m-n) \, \tfrac{1}{4} k^3y^3 (1 + \tfrac{3}{4} k^2y^2 + 9/16 \, k^4y^4 - \frac{1}{\sqrt{4}} (\cos 4 (m-n) - \sin 4 (m-n)) \, \tfrac{1}{8} k^4y^4 (1 + 4^4 k^2y^2 + 4/16 \, k^4y^4 + \frac{1}{\sqrt{5}} (\cos 5 (m-n) - \sin 5 (m-n)) \, 1/16 \, k^5y^5 (1 + 5/4 \, k^2y^2 + \ldots . - \ldots .) \Big] \qquad (11)$$

References:

1. Carson, J. R., Notes on the theory of modulation. Proc. I. R. E. Feb., 1922, page 57.
2. Armstrong, E. H., A Method of Reducing Disturbances in Radio Signalling by a System of Frequency Modulation. Proc. I. R. E. May 1936, page 689.
3. Roder, Hans. Amplitude, Phase, and Frequency Modulation. Proc. I. R. E. Dec. 1931, page 2145, Equation (22).