he response is flat to 0.06 db.

$f(0.15) = 0.125 \qquad f_2(0.15) = 0.061$

$f(0.10) = 0.127 \qquad f_2(0.10) = 0.085$

$f(0.65) = 0.376 \qquad f_2(0.65) = 0.112$

$$F(-d) = (4 \times 0.376 + 0.854)(4 \times 0.112 + 0.1465)x_0^4$$
$$= 2.359 \times 0.5945x_0^4 = 1.401x_0^4$$

$$F(-a) = (4 \times 0.125 + 0.854)(4 \times 0.061 + 0.1465)x_0^4$$
$$= 1.354 \times 0.3905x_0^4 = 0.529x_0^4$$

$$F(-b) = (4 \times 0.127 + 0.854)(4 \times 0.085 + 0.1465)x_0^4$$
$$= 1.362 \times 0.4865x_0^4 = 0.663x_0^4$$

$$\frac{F(-d) - F(-a)}{(d-a)} = \frac{1.401 - 0.529}{1.0}x_0^3 = 0.872x_0^3$$

$$\frac{F(-b) - F(-a)}{(b-a)}$$
$$= \frac{1}{2}(2)(0.25 - 0.293)(0.3905 + 0.4865)x_0^3$$
$$+ \frac{1}{2}(2)(0.25 - 0.707)(1.354 + 1.362)x_0^3$$
$$= -1.278x_0^3$$

$$K_{34}^4 - K_{34}^2x_0^2(0.872/1.1 + 0.1 \times 1.0)$$
$$+ x_0^4(1.401)(0.1)/(1.1) = 0$$

$K_{34}^2 = 0.7135x_0^2 \qquad$ $K_{34} = 0.845x_0 = 0.0425$

$$K_{12}^2 = x_0^2(0.529)/(0.7135 - 0.1)$$
$$= 0.862x_0^2$$

$K_{12} = 0.93x_0 = 0.0465$

$$K_{23}^2 = x_0^2(1.278 - 0.862 \times 1.0)/(1.1)$$
$$= 0.379x_0^2$$

$K_{23} = 0.615x_0 = 0.03075$

If $L_{22} = L_{33} = 5\ \mu$h,

$$C_{23} = (20 \text{ mmfd})/0.03075 = 650 \text{ mmfd}.$$

With $g_m = 5 \times 10^{-3}$ mhos

$$\text{Gain} = g_mX_cK_{12}K_{23}K_{34}(8\alpha/x_0^4)$$
$$= 5 \times 10^{-3} \times 1800 \times 0.93 \times 0.615$$
$$\times 0.845 \times 0.95/0.05 = 83.$$

At 14 Mc, the attenuation is $0.95 \times (0.267/0.05)^4 = 775$.

Conclusion

We have concerned ourselves, in this paper, with the design of multiple-tuned circuits for optimum amplitude response. One might think that better practical results would be obtained with flat designs. The opposite is actually the case. A small σ requires a large (M/x_0) and consequently large coefficients for $F(p)$, but all the coefficients of $F(p)F(-p)$ with the exception of the constant term are independent of σ. Thus the effect of reducing σ is to require that the differences of large numbers be small numbers.

The design equations for the electrical parameters are expressed in terms of the invariant $F(p)$, so that these equations apply in form for any desired response characteristics, it being only necessary to determine the appropriate $F(p)$.

Discussion on

"Properties of Some Wide-Band Phase-Splitting Networks"*

DAVID G. C. LUCK

Frederick E. Bond[1]: While reading the paper entitled "Properties of some wide-band phase-splitting networks," by David G. C. Luck, it appeared that there might be a simpler mathematical procedure for deriving the relationship between deviation from 90° phase difference and the ratio of the maximum and minimum frequency.

Specifically, consider equation (6) in the above-mentioned paper, which expresses the tangent of half the phase difference as a function of the circuit Q (as defined therein); the quantity r (the square root of the ratio of f_2 to f_1); and f_0 (the geometric mean of f_1 and f_2).

$$\tan \tfrac{1}{2}\psi = \frac{Q\left[\left(\frac{rf_0}{f} - \frac{f}{rf_0}\right) - \left(\frac{f_0}{rf} - \frac{rf}{f_0}\right)\right]}{1 + Q^2\left(\frac{rf_0}{f} - \frac{f}{rf_0}\right)\left(\frac{f_0}{rf} - \frac{rf}{f_0}\right)}.$$

The nature of this function suggests the following change of variables:

* Proc. I.R.E., vol. 37, pp. 147–152; February, 1949.
[1] Coles Signal Laboratory, Red Bank, N. J.

Let

$$f/f_0 = e^x \quad (1)$$

and

$$r = e^y. \quad (2)$$

Then the above expression becomes

$$\tan \tfrac{1}{2}\psi = \frac{4Q \sinh y \cosh x}{1 - 4Q^2(\cosh^2 y - \cosh^2 x)}. \quad (3)$$

Now when $f = f_0$, $x = 0$. Therefore

$$\tan \tfrac{1}{2}\psi_0 = \frac{4Q \sinh y}{1 - 4Q^2 \sinh^2 y}. \quad (4)$$

Combining (3) and (4)

$$\frac{\tan \frac{1}{2}\psi}{\tan \frac{1}{2}\psi_0} = \frac{\cosh x}{1 + M \sinh^2 x} \quad (5)$$

where

$$M = \frac{4Q^2}{1 - 4Q^2 \sinh^2 y} = \frac{4Q^2}{1 - Q^2\left(r - \frac{1}{r}\right)^2}. \quad (6)$$

Note that M is equal to $\sin^2 \frac{1}{2}\sigma$ where σ is defined in equation (11) in the paper under discussion. In equation (5) we have the phase difference expressed as a function of ψ_0 (its value at the mean frequency f_0); x which is equal to $\log_e f/f_0$; and M, which is a function of the circuit parameters Q and r.

In order to find the location of the maxima and minima for the ratio $\tan \frac{1}{2}\psi / \tan \frac{1}{2}\psi_0$, setting the first derivative of equation (5) equal to zero yields

$$\sinh x = 0, \quad \pm \sqrt{\frac{1 - 2M}{M}}. \quad (7)$$

Equation (7) shows immediately that in order to realize the double humps in the curves shown in Fig. 4 of the article in question, M must be less than 1/2 and greater than zero.

To find the values of $\tan \frac{1}{2}\psi_m$, substituting (7) in (5) yields

$$\frac{\tan \frac{1}{2}\psi_m}{\tan \frac{1}{2}\psi_0} = \frac{1}{2\sqrt{M(1 - M)}}. \quad (8)$$

Now to determine the maximum deviation of phase difference from the average value,

$$\frac{\sin \frac{1}{2}(\psi_m - \psi_0)}{\sin \frac{1}{2}(\psi_m + \psi_0)} = \frac{\dfrac{\tan \frac{1}{2}\psi_m}{\tan \frac{1}{2}\psi_0} - 1}{\dfrac{\tan \frac{1}{2}\psi_m}{\tan \frac{1}{2}\psi_0} + 1} = \frac{1 - 2\sqrt{M(1 - M)}}{1 + 2\sqrt{M(1 - M)}}. \quad (9)$$

For the quadrature case, then $1/2(\psi_m + \psi_0) = 90°$ and

$$\sin \tfrac{1}{2}(\psi_m - \psi_0) = \frac{1 - 2\sqrt{M(1 - M)}}{1 + 2\sqrt{M(1 - M)}}. \quad (10)$$

The find f_{min} and f_{max}, i.e., the frequencies at which $\tan \frac{1}{2}\psi = \tan \frac{1}{2}\psi_0$, thus representing the maximum band spread for a deviation of $\frac{1}{2}(\psi_m - \psi_0)$, equate (5) to unity. Then

$$\cosh x = 1, \quad \frac{1}{M} - 1. \quad (11)$$

The first value of course refers to $f = f_0$ where $x = 0$. The second value refers to both f_{max} and f_{min}, since cosh x is an even function of x and x has logarithmic symmetry with f.

From (11)

$$e^x = \frac{f}{f_0} = \frac{1 - M \pm \sqrt{1 - 2M}}{M}. \quad (12)$$

The expression in (12) with the positive sign must refer to f_{max} since it is always greater than the value using the negative sign. Therefore

$$\frac{f_{max}}{f_{min}} = \frac{\dfrac{f_{max}}{f_0}}{\dfrac{f_{min}}{f_0}} = \frac{1 - M + \sqrt{1 - 2M}}{1 - M - \sqrt{1 - 2M}}. \quad (13)$$

Using corresponding values of M in equations (10) and (13), the desired relationship between maximum deviation and frequency spread can be obtained and the values in Fig. 6 of Dr. Luck's paper can be checked.

David G. C. Luck[2]: The interest which has led Mr. Bond to work out an alternative to the methods used in my paper is greatly appreciated.

Introducing the substitute variables η and σ, by equations (9) and (11) of my paper, I stated that this was done quite arbitrarily. The same statement applies also to the introduction of θ and ρ by my equations (13) and (18). Various other arbitrary substitutions could, of course, have been used instead.

Mr. Bond chooses to use cosh x wherever I used cosec η, $\sqrt{M}$ where I used $\sin \frac{1}{2}\sigma$, and sinh γ where I used cot ρ. These seem to me to be purely matters of personal preference. There is no need to justify by logic either his preference or mine. Mr. Bond also chooses to stop short of my final substitution (13), which in his notation would be

$$\tan \tfrac{1}{2}\theta = \sqrt{\frac{M}{1 - M}} \cosh x. \quad (A)$$

[2] RCA Laboratories Division, Princeton, N. J.

is final expression (5) for the phase difference-versus-equency characteristic is essentially my equation (12), th x and M substituted for my η and σ. This is again rgely a matter of preference and, as Mr. Bond has own, the essential results can be derived readily from s expression.

It is certainly Mr. Bond's right to make his calculaons in any self-consistent way he chooses. However, cannot quite agree that it is simpler to determine aximum and minimum properties by the formal ocesses of taking derivatives, equating them to zero, id substituting back the results, than to determine iese properties by inspection, in the light of common nowledge of the shapes of trigonometric functions. either can I agree that it is simpler to compute nuerical results from Mr. Bond's quadratic expressions 5), (7), (8), (9), and (10) than to pick such results ady-made from the nearest trig table, with the aid of y expressions (13), (14), (15), and (16).

Unfortunately, I have not been able to find any really onvenient expression for f_{max}/f_{min}; for comparison with Ir. Bond's equation (13), my notation offers,

$$\frac{f_{max}}{f_{min}} = \left(\frac{1+\sqrt{\cos\sigma}}{1-\sqrt{\cos\sigma}}\right)^2. \tag{B}$$

Also, for slide-rule computation, my expressions (17) and (18) can be written as

$$Q = \frac{\sqrt{\cos\frac{1}{2}\psi_0}}{2\cos\frac{1}{4}\psi_0}\sin\tfrac{1}{2}\sigma \tag{C}$$

and

$$\tan\rho = \frac{\sqrt{\cos\frac{1}{2}\psi_0}}{\sin\frac{1}{4}\psi_0}\sin\tfrac{1}{2}\sigma. \tag{D}$$

The intermediate variables η and σ (or x and y), which disappear from my final expressions, have the virtue of avoiding quadratic solutions in the inverse calculation of frequency ratios from desired phase-difference characteristics. If one accepts computation of quadratic forms anyway, Mr. Bond's x and y seem to lose much of their utility. My introduction of η was based, however, on the further supposition that the graphical form of a cosecant is more generally recognized than that of $f/f_0+f_0/f$, which cosec η replaces. The same justification might apply to Mr. Bond's use of cosh x, if he wishes to derive results, by inspection, from the graphical shapes of known functions.

CORRECTION

Albert S. Richardson, Jr., author of the paper "The remainder theorem and its applications to operational calculus techniques," which appeared on pages 1336–1339 of the November, 1950, issue of the PROCEEDINGS OF THE I.R.E., has brought the following errors to the attention of the editors:

On page 1337, the last three terms of equation (4) should read

$$K_1e^{S_{1q}t} + K_2e^{S_{2q}t} + \cdots + K_qe^{S_{qq}t}$$

instead of

$$K_1e^{S_{1q}t} + K_2e^{S_{2q}t} + \cdots + K_qe^{S_{qq}t}.$$

Also on page 1337, the right-hand side of the last equation on this page should read

$$[e^{+S_{qq}t} - 1]$$

instead of

$$[e^{-S_{qq}t} - 1].$$