Equation (10) becomes larger than (9) if $m \geqq 2$ because

$$2 \cdot 2^{n/2+2} > 2 \cdot 3 \cdot 2^{n/2}. \quad (11)$$

Next, consider the case where n is an odd number. If the most nearly even division is made,

$$E(n) = 2 \cdot 2^n + E\left(\frac{n+1}{2}\right) + E\left(\frac{n-1}{2}\right)$$
$$\leqq 2 \cdot 2^n + 2 \cdot 3 \cdot 2^{(n+1)/2}. \quad (12)$$

If a less even division is made,

$$N(n) = 2 \cdot 2^n + E\left(\frac{n+1}{2} + m\right) + E\left(\frac{n-1}{2} - m\right)$$
$$> 2 \cdot 2^n + 2 \cdot 2^{(n+1)/2+m}. \quad (13)$$

Equation (13) is larger than (12) when $m \geqq 2$ because

$$2 \cdot 2^{(n+1)/2+2} > 2 \cdot 3 \cdot 2^{(n+1)/2}. \quad (14)$$

In order to proceed with the proof and demonstrate that $(m=0)$ is better than $(m=1)$, rather delicate tests must be applied. Consider four cases:

Let

$$n = 4p. \quad (15)$$

For the even division write

$$E(4p) = 2 \cdot 2^{4p} + 2 \cdot 2 \cdot 2^{2p} + 4E(p); \quad (16)$$

while the odd division gives

$$N(4p) = 2 \cdot 2^{4p} + E(2p+1) + E(2p-1) \quad (17)$$
$$N(4p) = 2 \cdot 2^{4p} + 2 \cdot 2^{2p+1} + 2 \cdot 2^{2p-1} + E(p+1) + 2E(p) + E(p-1); \quad (18)$$

Writing the appropriate inequalities for (16) and (18),

$$E(4p) \leqq 2 \cdot 2^{4p} + 2 \cdot 2 \cdot 2^{2p} + 2E(p) + 2 \cdot 3 \cdot 2^p \quad (19)$$
$$N(4p) > 2 \cdot 2^{4p} + 2 \cdot 2 \cdot 2^{2p} + 2E(p) + 2^{2p} + 2 \cdot 2 \cdot 2^p + 2 \cdot 2^{p-1}. \quad (20)$$

It is then obvious that

$$E(4p) < N(4p) \quad \text{when} \quad p \geqq 1. \quad (21)$$

The same procedure may be applied when $n=4p+1$, $4p+2$, and $4p+3$, but the details will not be presented since the method has been demonstrated.

Properties of Some Wide-Band Phase-Splitting Networks*

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Summary—**Passive networks that produce polyphase output from single-phase input over wide frequency bands have great practical utility, yet published analysis of their action is incomplete. After reviewing properties of simple branch circuits useful in such networks, an expression is derived for phase difference produced between branches as a function of frequency. This expression is given a remarkably simple form, from which over-all operating properties are evident by inspection, permitting direct circuit design from required performance. General performance and design curves are presented.**

Introduction

EVERY RADIO ENGINEER has at some time produced two voltages in known phase relation, probably by use of a simple RC phase splitter, and many have been annoyed by the sensitivity of such devices to frequency changes. Although twenty years have gone by since Zobel[1] described some phasing circuits with constant attenuation, the possibility of using such circuits to build up wide-band phase-splitting networks has remained very little known. A recent paper by Dome[2] has therefore performed a valuable service by calling general attention to the possibility of thus making passive networks which will develop a polyphase signal from a single-phase source having complex wave form or variable frequency. He has also pointed out the great utility of such devices for single-sideband modulation, frequency-shift keying, development of circular oscilloscope sweep, and like applications.

Because of their wide applicability, the properties of these circuits deserve a more complete analysis than Dome has published. Such analysis turns out to allow the performance attainable to be displayed in a convenient and general fashion, and in addition to permit direct determination of circuit parameters from required performance. The most elementary sort of analysis will provide all the information needed, so long as a tapped signal source having negligible internal impedance can be assumed available. The properties of some simple branch circuits that may be used will first be indicated as briefly as possible, to provide a foundation for the analysis of the complete phase-splitting network which follows. The general properties of the complete circuit are then discussed and presented graphically, and the results attainable in the special case of a 90° phase splitter are indicated.

Properties of Branch Circuits

The very simple series circuit of Fig. 1(a) exhibits all essential properties, but its output must not be loaded. This circuit, with only three elements, is fully deter-

* Decimal classification: R143×R246.2. Original manuscript received by the Institute, April 16, 1948; revised manuscript received, June 17, 1948. Presented, 1948 IRE National Convention, March 22, 1948, New York, N. Y.

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[1] O. J. Zobel, "Distortion correction in electrical circuits," *Bell Sys. Tech. Jour.*, vol. 7, pp. 438–534; July, 1928.

[2] R. B. Dome, "Wide band phase shift networks," *Electronics*, vol. 19, pp. 112–115; December, 1946.

mined by specifying its operating characteristics of resonant impedance R, resonant frequency f_1, and selectivity Q (reactance/resistance ratio at resonance). Output voltage e is evidently related to input voltage E by

$$e = -\tfrac{1}{2}E + \frac{R}{R + j\omega L + \dfrac{1}{j\omega C}} E, \qquad (1)$$

considering the circuit as merely a voltage divider.

Introducing the operating characteristics f_1 and Q into (1), the complex voltage-transfer coefficient e/E becomes in general

$$e/E = k\frac{1 + jQ(f_1/f - f/f_1)}{1 - jQ(f_1/f - f/f_1)}, \qquad (2)$$

with the real amplitude factor k independent of frequency and having just the value $\tfrac{1}{2}$ for the particular circuit in question. Now,

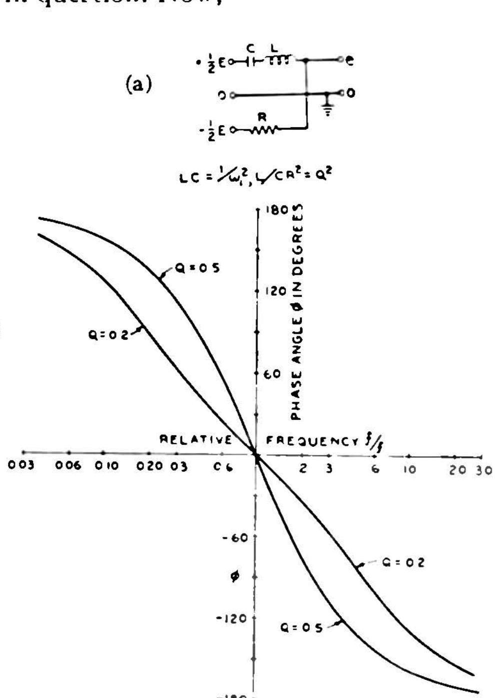


Fig. 1—Simple branch circuit and its phase characteristic.

$$Q(f_1/f - f/f_1) = \tan \tfrac{1}{2}\phi \qquad (3)$$

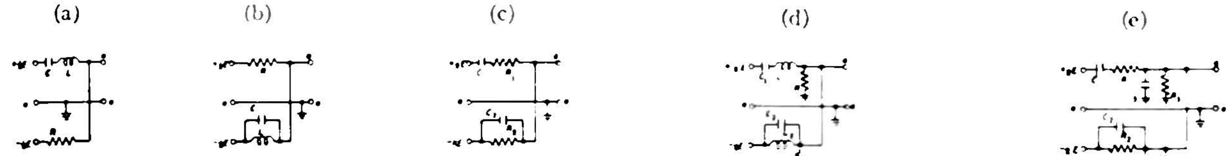
where $\tfrac{1}{2}\phi$ is the leading phase angle of the current in the series circuit. In terms of this angle, the voltage transfer is simply

$$e/E = k\epsilon^{j\phi}. \qquad (4)$$

The relative phase angle ϕ of the output voltage is plotted in Fig. 1(b) against f/f_1, on a logarithmic frequency scale, for two representative values of Q. At the higher value of Q shown, the curve has a simple S shape, with a single point of inflection. At the lower value of Q, the curve has a double-S shape, with three points of inflection. At an intermediate value of Q, there is a transition between these two limiting forms, and the corresponding curve is substantially straight over a considerable region. In any case, ϕ approaches $\pm 180°$ for frequencies far from resonance.

The properties of the circuit that are important to the present discussion are the constant magnitude k of input to output voltage-transfer ratio and the fairly linear region of the phase-frequency curve that is symmetrically located around its point of inflection at resonance. It is also important to remember that the shape of the phase curve is completely determined by specifying Q. On a logarithmic frequency scale, the resonant frequency f_1 merely locates the curve without altering its form, and resonant impedance R does not directly affect the transfer characteristics at all.

Several alternative circuits capable of giving the desired characteristics are shown in Fig. 2, with relations between values of circuit elements and f_1, Q, and k shown for each. Q, used here for its familiar significance, is just the reciprocal of the parameter s of Dome's notation. Since the resistance R is external to the resonant loop in Fig. 2(b), the Q in that case is the ratio of resistance to reactance of one element at resonance. Values of Q that are of present interest are less than $\tfrac{1}{2}$, so circuits using only one kind of reactance are permissible, if the penalty of sharply reduced output can be tolerated. The fourth element of Fig. 2(c) is not determined by f_1, Q, and over-all impedance required, and may be ad-



(a)
$\omega_1^2 = 1/LC$
$Q^2 = L/CR^2$
$k = \tfrac{1}{2}$
$L = QR/\omega_1$
$C = 1/\omega_1 QR$

(b)
$\omega_1^2 = 1/LC$
$Q^2 = CR^2/L$
$k = \tfrac{1}{2}$
$L = R/\omega_1 Q$
$C = Q/\omega_1 R$

(c)
$\omega_1 = 1/R_1C_1 = 1/R_2C_2$
$Q = 1/[R_2/R_1 + 2]$
$k = 1 - 2Q$
$R_1 = R_2/[1/Q - 2]$
$C_2 = 1/\omega_1 R_2$
$C_1 = [1/Q - 2]/(\omega_1 R_2)$

(d)
$\omega_1^2 = 1/L_1C_1 = 1/L_2C_2$
$Q^2 = L_1/C_1R^2 = C_2R^2/L_2$
$k = \tfrac{1}{2}$
$L_1 = QR/\omega_1$
$C_1 = 1/\omega_1 QR$
$L_2 = R/\omega_1 Q$
$C_2 = Q/\omega_1 R$

(e)
$\omega_1 = 1/R_1C_1 = 1/R_2C_2 = 1/R_3C_3$
$Q = 1/[R_2/R_1 - 2] = 1/[R_3/R_2 + 2]$
$k = (1 - 2Q)/(1 + 2Q)$
$R_2 = R_3/[1/Q - 2]$
$R_1 = R_3/[1/Q^2 - 4]$
$C_3 = 1/\omega_1 R_3$
$C_2 = [1/Q - 2]/(\omega_1 R_3)$
$C_1 = [1/Q^2 - 4]/(\omega_1 R_3)$

Fig. 2—Alternative branch circuits and their properties.

justed to give maximum k for a chosen Q, a result obtained by making the two RC products equal.

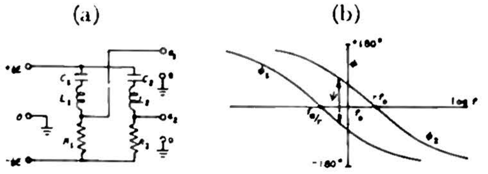
Fig. 2(d) shows one-half of the conventional symmetrical phase-compensating lattice, which is able to tolerate the resistive load $\frac{1}{2}R$ at the half-lattice output. In order to produce the desired transfer characteristics as given by (2) or (4), resonant frequencies and Q values must be made the same for the two branches shown. When these conditions are imposed, the circuit elements are fully determined and, in addition, the full lattice section exhibits a purely resistive input impedance R which is independent of frequency.

Dome[2] has also suggested the very useful circuit of Fig. 2(e), which not only avoids the use of inductances but will tolerate a load that includes shunt capacitance. This circuit is fully determined and made to follow (2) by imposing three conditions, which equalize the values of f_1 and Q for the two branches and maximize k. All circuits shown except that of Fig. 2(c) are half lattices fed in balanced fashion, and may be expanded by symmetry to the full lattice section, if a balanced output of doubled amplitude is desired.

Analysis of Complete Circuit

Any of the circuits of Fig. 2, all of which are characterized by attenuation independent of frequency and an output phase varying as in Fig. 1(b), can be used as elements of a phase splitter. If two such circuits of equal Q are tuned to different frequencies f_1 and f_2 and connected as in Fig. 3(a) to provide separate branch-output voltages e_1 and e_2, the phase relations of Fig. 3(b) will result. The mean phase of e_1 and e_2 relative to E will vary with frequency much as do the phases ϕ_1 and ϕ_2 of e_1 and e_2 separately. On the other hand, the phase difference ψ between e_1 and e_2 will take on a fairly uniform maximum value for frequencies near the mean of f_1 and f_2, where the individual phase curves are fairly linear, and will approach zero for frequencies far above f_2 or below f_1.

(a) (b)



$\omega_1^2=\omega_0^2/r^2=1/L_1C_1$ $\omega_2^2=\omega_0^2r^2=1/L_2C_2$ $Q^2=L_1/C_1R_1^2=L_2/C_2R_2^2$

Fig. 3—Complete phase-splitter circuit and its branch-phase characteristics.

The behavior of the complete phase splitter is fully determined by the resonant frequencies f_1 and f_2, together with the common value of Q. Alternatively, the geometric mean f_0 between f_1 and f_2 and the ratio r^2 of f_2 to f_1 are convenient to use as basic parameters. In conjunction with any conditions necessary to provide the characteristic of (2), and to maximize k, the three parameters f_0, r, and Q also determine the values of all circuit elements except as regards over-all impedance level. The relation between branch-output phase difference ψ and frequency f may now be examined in detail as to its general form and its dependence on the parameters f_0, r, and Q.

Since f_1 is by definition f_0/r and f_2 is rf_0, the phase angles ϕ_1 and ϕ_2 of output voltages e_1 and e_2 relative to input voltage E are given respectively by

$$\left.\begin{aligned}\tan\tfrac{1}{2}\phi_1 &= Q\left(\frac{f_0}{rf}-\frac{rf}{f_0}\right)\\ \text{and}\qquad \tan\tfrac{1}{2}\phi_2 &= Q\left(\frac{rf_0}{f}-\frac{f}{rf_0}\right)\end{aligned}\right\}. \tag{5}$$

The phase lead ψ of e_2 over e_1 is $\phi_2-\phi_1$, and use of (5) in the usual trigonometric expression for the tangent of the difference of two angles, gives directly

$$\tan\tfrac{1}{2}\psi = \frac{Q\left[\left(\frac{rf_0}{f}-\frac{f}{rf_0}\right)-\left(\frac{f_0}{rf}-\frac{rf}{f_0}\right)\right]}{1+Q^2\left(\frac{rf_0}{f}-\frac{f}{rf_0}\right)\left(\frac{f_0}{rf}-\frac{rf}{f_0}\right)}. \tag{6}$$

Expanding this expression, factoring the numerator, rearranging the denominator to give perfect squares of factors in the numerator, and dividing through by Q^2 gives in turn

$$\tan\tfrac{1}{2}\psi = \frac{\frac{1}{Q}\left(r-\frac{1}{r}\right)(f/f_0+f_0/f)}{\frac{1}{Q^2}-(r-1/r)^2-4+(f/f_0+f_0/f)^2} \tag{7}$$

as a general expression for the frequency dependence of the phase split produced by any equal-Q pair of circuits of the type considered here.

Equation (7), with evident logarithmic symmetry about the central frequency f_0, is already quite simple and usable. Certain changes of variable render its form much simpler still, however, and in addition permit almost all numerical calculations of performance to be taken ready-made from trigonometric tables. Simplification is accomplished by use of the standard trigonometric identities

$$\left.\begin{aligned}\cot x+\tan x &= \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x\\ \text{and}\qquad \cot x-\tan x &= \frac{2}{\tan 2x} = 2\cot 2x\end{aligned}\right\}. \tag{8}$$

Quite arbitrarily, let

$$f/f_0 = \tan\tfrac{1}{2}\eta. \tag{9}$$

$$Q\left(r-\frac{1}{r}\right) = \tan\tfrac{1}{2}\psi_0, \tag{10}$$

and

$$\frac{1}{Q^2}-\left(r-\frac{1}{r}\right)^2 = 4\operatorname{cosec}^2\tfrac{1}{2}\sigma \tag{11}$$

define a new independent variable η and new parameters ψ_0 and σ. Substituting these variables into (7) and using (8), there results

$$\tan\tfrac{1}{2}\psi = \tan\tfrac{1}{2}\psi_0\,\frac{\operatorname{cosec}^2\frac{1}{2}\sigma\operatorname{cosec}\eta}{\cot^2\frac{1}{2}\sigma+\operatorname{cosec}^2\eta}. \tag{12}$$

Now let

$$\tan \tfrac{1}{2}\sigma \operatorname{cosec} \eta = \tan \tfrac{1}{2}\theta \qquad (13)$$

define still another independent variable θ. In terms of this variable, (12) [and thereby (7)] finally takes the form

$$\frac{\tan \frac{1}{2}\psi}{\tan \frac{1}{2}\psi_0} = \frac{\sin \theta}{\sin \sigma} \cdot \qquad (14)$$

This is the general phase-versus-frequency characteristic of the complete network. It holds for any values of f_0, r, and Q, but these have now been replaced by more convenient performance parameters ψ_0 and σ.

Properties of Phase Splitter

The physical significance of the above very compact result shows quite clearly upon examination of (13) and (14), bearing (9) also in mind, and determines the form of the curves to be given later. For an input-signal frequency equal to the network center frequency f_0, η is 90 degrees and cosecη is unity so, from (13), θ is equal to σ and, from equation (14), branch-phase difference ψ has just the value ψ_0. That is, ψ_0, determined by Q and r according to (10), turns out to be just the phase split between the two branch outputs for the center frequency f_0.

Since cosecη cannot be less than unity, (13) indicates that θ cannot be less than σ. If σ exceeds 90°, $\tan\frac{1}{2}\sigma$ exceeds unity and θ remains above 90° for all frequencies. Equation (14) then indicates a single phase-difference maximum when θ, still above 90°, is a minimum. θ is a minimum when cosecη has its minimum of unity, at frequency f_0. The network with $\sigma > 90°$ thus exhibits a single maximum of phase difference, having the value ψ_0, at the center frequency f_0.

If, on the other hand, σ is less than 90°, $\tan\frac{1}{2}\sigma$ is less than unity and θ will pass through 90° as cosec η takes the value $\cot\frac{1}{2}\sigma$ (>1) in its variation with frequency. Since η varies from 0° to 180° as input frequency goes from 0 to ∞, cosecη will take this value twice and phase difference ψ will show in this case two maxima, of value ψ_m. These maxima will be symmetrically located with respect to f_0, since cosecη is symmetrical in $\log f/f_0$. At f_0, θ will have its minimum value, less than 90° and equal to σ, and the phase difference ψ will now exhibit at f_0 a minimum of value ψ_0.

When there is a double maximum of phase difference, the relation

$$\operatorname{cosec} \sigma = \frac{\tan \frac{1}{2}\psi_m}{\tan \frac{1}{2}\psi_0} \qquad (15)$$

holds and, alternatively, the departure $\frac{1}{2}(\psi_m - \psi_0)$ of the extremes of phase difference from the mean $\frac{1}{2}(\psi_m + \psi_0)$ of those extremes is determined by σ as

$$\sin \tfrac{1}{2}(\psi_m - \psi_0) = \sin \tfrac{1}{2}(\psi_m + \psi_0) \tan^2 (45° - \tfrac{1}{2}\sigma). \qquad (16)$$

For values less than 90°, the physical meaning of σ is thus quite simple and clear.

Parameters f_0, ψ_0, and σ determine the relation between branch-output phase difference and signal frequency completely and in a very simple and direct way. Center frequency f_0 locates the characteristic on a logarithmic frequency scale, without affecting its form. Center-frequency phase difference ψ_0 locates the characteristic on a logarithmic-tangent phase scale without affecting its form. Form parameter σ controls the general shape of the characteristic and its spread in frequency, as well as further locating it on the phase scale. A single characteristic curve, generally useful for all values of f_0 and ψ_0, is therefore obtainable for each value of σ.

The parameters f_0, ψ_0, and σ also determine the circuit design quite directly. From (10) and (11) together,

$$Q = \tfrac{1}{2}\sqrt{1 - \tan^2 \tfrac{1}{4}\psi_0} \sin \tfrac{1}{2}\sigma. \qquad (17)$$

Letting

$$r = \cot \tfrac{1}{2}\rho, \qquad (18)$$

(8), (10), and (11) give

$$\tan \rho = \sqrt{\cot^2 \tfrac{1}{4}\psi_0 - 1} \sin \tfrac{1}{2}\sigma. \qquad (19)$$

Resonant frequencies f_1 and f_2 for the two branch networks are f_0/r and rf_0, respectively, and the values of all circuit elements may be determined from f_1, f_2, Q, and a choice of impedance level.

Fig. 4 shows the phase-frequency characteristics, as given by joint use of (9), (13), and (14), for various

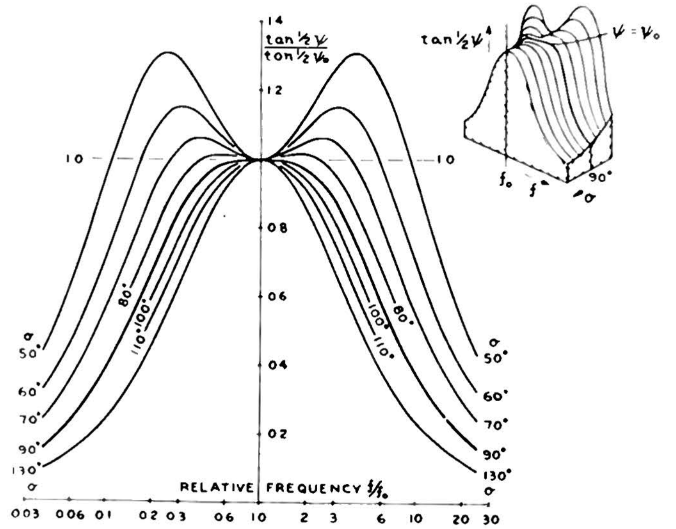


Fig. 4—General characteristics of phase splitter.

values of σ. Frequency is plotted logarithmically as f/f_0, and phase linearly as

$$\frac{\tan \frac{1}{2}\psi}{\tan \frac{1}{2}\psi_0} \cdot$$

Curves for ψ directly would look much the same, but would change shape with changing choice of ψ_0; those shown are more general. The perspective sketch shows the over-all relationship of ψ to f and σ more clearly, though less quantitatively, than does its projection as a family of curves in the f–ψ plane. Values of σ exceeding 90° lead to rapidly varying phase difference and are not of great interest. For σ just 90°, there is a truly flat region just at f_0, where ψ_0 and ψ_m are the same. This is a useful condition for purposes which require accurately constant phase difference over a narrow frequency band, and corresponds to

$$Q = \tfrac{1}{2}\sqrt{\tfrac{1}{2}(1 - \tan^2 \tfrac{1}{4}\psi_0)}. \quad (20)$$

For σ less than 90°, the characteristic becomes double-humped, with maxima equal to cosec σ and a minimum of unity on the relative half-angle tangent scale used. These maxima occur at frequencies for which $\sin\eta$ is just $\tan\frac{1}{2}\sigma$. Phase difference again falls to ψ_0 at the two frequencies, $f_{\min}$ and $f_{\max}$, outside those at the phase maxima, for which $\sin\eta$ becomes $\tan^2\frac{1}{2}\sigma$. Values of less than 90° for σ lead to the conditions of most practical interest, for which phase difference remains close to a specified value over a considerable band of frequencies. When ψ_m and ψ_0 are specified, σ is given by (15) or (16) and the frequencies at which ψ_m and return to ψ_0 occur are fully determined, as are Q and r. Actual spread $\psi_m - \psi_0$ of the phase difference produced depends on the choice of both ψ_0 and σ, as do the circuit defining parameters r and Q. The ratio of maximum to minimum frequency of the working band over which ψ exceeds ψ_0 depends only on σ, however.

More detailed investigation shows that, for a given total spread of phase angle, a wider frequency band is obtained by choosing a value of σ which places ψ_m at the upper limit and ψ_0 at the lower limit of the permitted phase spread than by choosing σ to place ψ_m at the upper limit and ψ_0 at, for example, the center of the spread. If a given phase split is to be approximated over a given frequency range, requiring a definite value of σ, less phase spread is encountered when a single network section with a large value of ψ_0 is used than when several isolated or iterative sections with equal small values of ψ_0 are used in cascade. The present analysis has not been extended to such single-section networks, of more complicated structure than those shown in Fig. 2, as are necessary to give still better constancy of branch-output phase difference over the working frequency band. Extension to more than two branch outputs, considered two by two, is obvious.

Quadrature Case

The case of 90° phase difference, corresponding to production of symmetrical two-phase output, is of particular importance and will be examined further. Fig. 5 shows operating characteristics $\frac{1}{2}(\psi_m - \psi_0)$ and $f_{\max}/f_{\min}$, as well as design parameters Q and r, plotted against the form parameter σ. $\frac{1}{2}(\psi_m + \psi_0)$ is here held at 90° for σ less than 90°; ψ_0 is held at 90° for σ greater than 90°. $\frac{1}{2}(\psi_m - \psi_0)$ is the maximum departure of phase difference from 90° within the frequency range from $f_{\min}$ to $f_{\max}$. The flat-topped phase characteristic for which $\psi_m = \psi_0 (\sigma = 90°)$ occurs in this case at a Q of 0.322, with r set at 1.835. Phase-difference characteristics with a single peak occur for all higher Q values, becoming increasingly sharp (σ larger) as Q increases. Double-

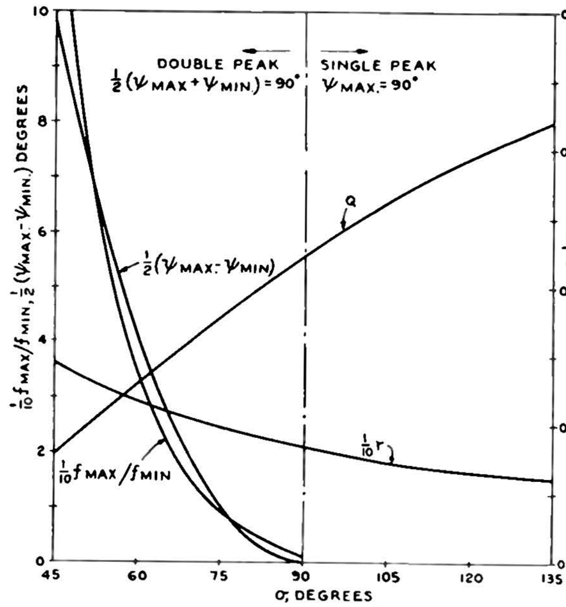


Fig. 5—Design characteristics of 90° phase splitter.

peaked characteristics occur only for Q values below 0.322, more pronouncedly as σ and Q decrease.

Fig. 6 summarizes the over-all performance of the 90° phase splitter with form parameter σ less than 90°, showing the relation between maximum error in phase difference produced in the working band and width of

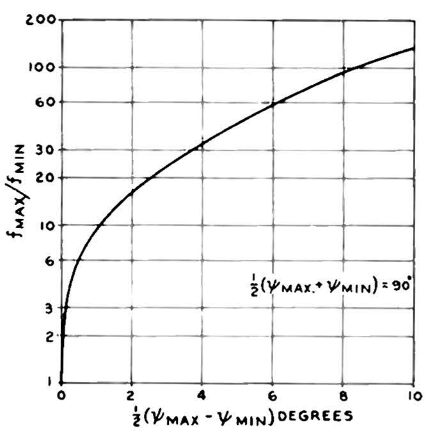


Fig. 6—Performance of 90° phase splitter.

working frequency band over which ψ remains above ψ_0. Operation over a frequency range of one full octave may in principle be had with branch-phase difference remaining within ±45 seconds of arc from 90°. The limiting frequencies are in a $9\frac{1}{2}$-to-1 ratio for operation within ±1° of 90°, while tolerance of ±5° variation of phase difference permits operation over a $43\frac{1}{2}$-to-1 band, which is as broad as the usual home-receiver audio band. Imperfections of actual circuit elements and their adjustments always degrade to some extent this ideal performance, however.

Evidently, these very simple circuits are capable ideally of very good performance, and their performance may be expressed in very convenient analytical form; this is the reason that the present study of their properties has been considered worth while.