The Design of Wide-Band Phase Splitting Networks*

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Summary—A number of articles and patents dealing with the properties and design of phase splitting networks, particularly in conjunction with single sideband modulators, have been published in the last few years. However, all of them have been restricted either to particular methods of design or to a particular number of design parameters. The present paper gives the results of a general investigation of phase splitting networks, dealing separately with network analysis, network synthesis and performance curve approximation problems. For the most important types of curve approximation, Taylor and Tchebycheff approximations, explicit formulas for any number of design parameters and for any required closeness of approximation are stated. Alternatives to the classical all-pass lattice network are given, and dissipation compensated phase shift networks are developed. In this way, clear and comparatively simple design instructions for simple as well as for difficult specifications for phase splitting networks are obtained. Furthermore, it is believed that some of the theoretical results obtained and methods developed, e.g., the Taylor and Tchebycheff approximations, the method of obtaining dissipation compensation, one of the methods of network synthesis, and the representation of approximating curves as iterated functions of two variables, with fractional index of iteration, are novel and of general theoretical interest.

I. INTRODUCTION

THE DESIGN of phase splitting circuits to produce constant phase differences over wide frequency bands has frequently been discussed during the last decade. Such circuits have chiefly been used for single sideband modulators for carrier telephony, for polyphase radio systems, for frequency-shift keying, and for wide-band circular time bases for cathode-ray oscillographs.

It is interesting to note that in three early references to phase splitting of a signal band no wide-band network for direct phase splitting is provided, but an auxiliary two-phase single frequency carrier supply with suitable modulator and demodulator stages is used instead,[1] (Wirkler,[2] Vilbig[3]). Hartley,[4] as early as 1925, described a wide-band phase splitting network consisting of two filters with different cutoff frequencies and different numbers of sections. Very simple wide-band phase splitting circuits which do not provide a constant amplitude output have been described by Honnell[5] and Lenehan.[6]

Wide-band phase splitting circuits consisting of two phase shifting networks with constant output amplitudes have been described by Byrne,[7] Loyet,[8] Hodgson,[9] Dome,[10] Norgaard,[11] and Luck.[12]

Byrne and Loyet give the theoretical and measured performance of various phase splitting circuits, but they do not give any design information. Hodgson, on the other hand, discusses design methods in detail. His main recommendation is to design each phase shift network separately so that its phase shift β over the frequency band in question varies linearly with the logarithm of the frequency, say $\beta = A + A_0 \log f$ where A and A_0 are constants. If A_0 is made the same for both networks but A is different for each network, say A_1 and A_2, then the difference of the two phase shifts β_1 and β_2 is $\beta_1 - \beta_2 = A_1 - A_2$, i.e., a constant, as required. Dome follows the same general idea, but whereas Hodgson's discussion is chiefly in terms of lattice and bridged-T phase shift networks, Dome describes a number of interesting alternatives to the classical lattice network. In Hodgson's patent the individual performances of the two phase shift networks are specified separately; Luck discusses the design and performance of a phase splitting circuit as a whole. This constitutes an important step forward. However Luck considers networks with four design parameters only.

It is the purpose of this paper to investigate phase splitting networks with any number of design parameters, for any desired bandwidth and for any desired closeness to the desired ideal performance.[13] This will be done in the following order: (1) network analysis, (2) performance curve approximation, (3) network synthesis. It will be found that it is comparatively easy to obtain results of general validity and applicability. In many respects the problems to be solved are similar to or identical with those encountered in the development of a comprehensive method and theory of filter design. However, in the case of phase splitting circuits consisting of constant resistance phase shift circuits the solution of these problems is easier than in the case of filter design, because complications due to mismatching do not arise. Furthermore, it so happens that in the theory

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[1] British Patent No. 301,362, dated August 27, 1927.
[2] Walter H. Wirkler, U. S. Patent No. 2,173,145, dated November 26, 1937.
[3] F. Vilbig, "Experimentelle Untersuchung der Verschiebung eines theoretisch beliebig grossen Frequenzbandes um einen bestimmten Phasenwinkel," *Telegraphen- Fernsprech- und Funktech.*, vol. 27, pp. 560–561; December, 1938.
[4] Ralph V. L. Hartley, U. S. Patent No. 1,666,206, dated January 15, 1925.
[5] M. A. Honnell, "Single-sideband generator," *Electronics*, vol. 18, pp. 166–168; November, 1945.
[6] B. E. Lenehan, "A new single sideband carrier system," *Elec. Eng.*, vol. 66, pp. 549–552; June, 1947.
[7] John F. Byrne, "Polyphase broadcasting," *Trans. Elec. Eng.*, vol. 58, pp. 347–350; July, 1939.
[8] Paul Loyet, "Experimental polyphase broadcasting," PROC. I.R.E., vol. 30, pp. 213–222; May, 1942.
[9] K. G. Hodgson, British Patent No. 547,601, dated January 31, 1941.
[10] R. B. Dome, "Wide-band phase shift networks," *Electronics*, vol. 19, pp. 112–115; December, 1946.
[11] Donald E. Norgaard, "A new approach to single sideband," *QST*, vol. 32, pp. 36–43; June, 1948.
[12] David G. C. Luck, "Properties of some wide-band phase splitting networks," PROC. I.R.E., vol. 37, pp. 147–151; February, 1949.
[13] Some of the networks obtained as a result of this investigation form the subject of British Patent Application No. 16698/49.

of the transformation of elliptic functions, which is applicable to filter as well as to phase splitting circuit design, the particular transformations applicable to phase splitting circuit design are simpler than those applicable to filter design.

II. The Basic Circuit

Fig. 1 shows schematically a phase splitting circuit consisting of two phase shift networks which are paralleled at their inputs. At this initial stage of our investigation the phase shift networks are assumed to be conventional all-pass constant resistance single lattice section networks with series arm reactances X_1 and X_2 and lattice arm reactances $-R_0^2/X_1$ and $-R_0^2/X_2$, respectively.

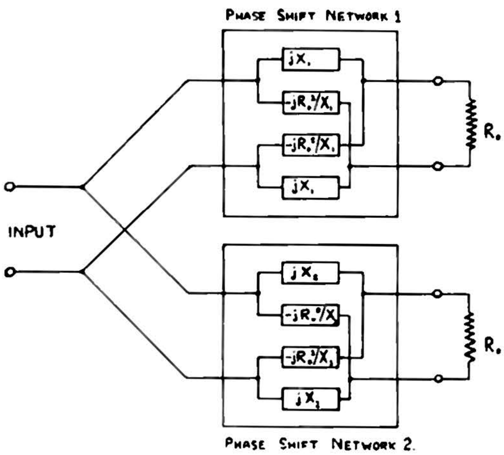


Fig. 1—Basic phase-splitting circuit.

Then the phase shifts β_1 and β_2 produced by the networks separately are given by

$$\tan \tfrac{1}{2}\beta_1 = X_1/R_0 \quad \text{and} \quad \tan \tfrac{1}{2}\beta_2 = X_2/R_0 \tag{1}$$

so that the phase shift difference $\psi = \beta_1 - \beta_2$ is given by

$$y = \tan \tfrac{1}{2}\psi = \tan \tfrac{1}{2}(\beta_1 - \beta_2) = \frac{X_1/R_0 - X_2/R_0}{1 + (X_1/R_0)(X_2/R_0)} \cdot \tag{2}$$

If $\beta_1 - \beta_2 = 90°$, $y = 1$. Thus in an ideal 90° phase splitting circuit, y should be unity, over a specified frequency range, or $|\log y| = 0$.

The significance of deviations of y from unity can only be discussed with reference to a particular application of the phase splitting circuit. It is interesting to consider a single sideband modulator using a phase splitting network. If it is assumed that all amplitude and phase relations are exactly as required (see Fig. 2), with the one exception that y is not exactly unity, it can be shown that the amplitude A_1 of the wanted sideband and the amplitude A_2 of the unwanted sideband, are given by

$$(L_1)_{db} = 20 \log_{10} |A_{10}/A_1| = 20 \log_{10} \sec \tfrac{1}{2}\delta$$
$$= 10 \log_{10} (1 + \tan^2 \tfrac{1}{2}\delta) \tag{3a}$$

$$(L_2)_{db} = 20 \log_{10} |A_{10}/A_2| = 20 \log_{10} \operatorname{cosec} \tfrac{1}{2}\delta$$
$$= 10 \log_{10} (1 + \cot^2 \tfrac{1}{2}\delta) \tag{3b}$$

where δ is the deviation of the phase difference $\beta_1 - \beta_2$ from 90°, i.e.,

$$\delta = \beta_1 - \beta_2 - \frac{\pi}{2} \tag{4}$$

and A_{10} is the value of A_1, for $\delta = 0$. If y is given, we can obtain δ directly from y, by means of

$$\tan \tfrac{1}{2}\delta = \frac{y-1}{y+1} \quad \text{or} \quad \sec^2 \tfrac{1}{2}\delta = \frac{2(y^2+1)}{(y+1)^2} \quad \text{or}$$
$$\operatorname{cosec}^2 \tfrac{1}{2}\delta = \frac{2(y^2+1)}{(y-1)^2} \cdot \tag{5}$$

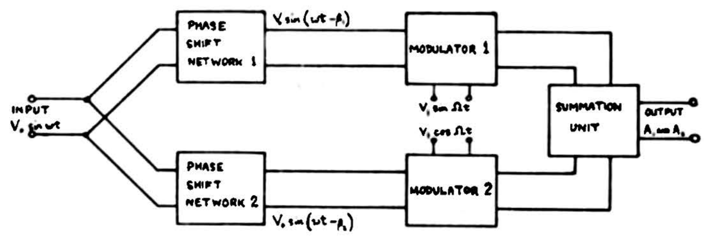


Fig. 2—Single sideband modulator using phase-splitting circuit.

Combining (3a) and (3b) with (5), we can obtain L_1 and L_2 as functions of y. The result has been plotted in Fig. 3. It is important to note that the transformation $y \to (1/y)$ leaves L_1 and L_2 unchanged and transforms δ into $-\delta$.

Equation (2) is very similar to an expression occurring in the evaluation of the insertion loss L of a lattice section filter between a source resistance R_0 and a load resistance R_0, with series arm reactances X_A and lattice arm reactances X_B. We find

$$L_{db} = 10 \log_{10} (1 + E^2) \tag{6a}$$

$$E = \frac{1 + (X_A/R_0)(X_B/R_0)}{(X_A/R_0) - (X_B/R_0)} \cdot \tag{6b}$$

It is seen by comparing (6b) and (2) that $1/E$ and y are formed in the same way from reactances X_A, X_B and X_1, X_2, respectively. This similarity has important consequences for the analysis and synthesis of phase splitting networks, and will be made use of in subsequent sections.

III. Network Analysis

The object of this section is to find, as a necessary preparation for network design and synthesis, the general characteristics of the function y defined by (2), if y is obtained from physical networks. Since

$$y = \tan \tfrac{1}{2}(\beta_1 - \beta_2) = \frac{\tan \frac{1}{2}\beta_1 - \tan \frac{1}{2}\beta_2}{1 + \tan \frac{1}{2}\beta_1 \tan \frac{1}{2}\beta_2}, \quad (7)$$

we start with a discussion of the characteristics of $\tan \frac{1}{2}\beta_1$ and $\tan \frac{1}{2}\beta_2$. From (1) it follows that $\tan \frac{1}{2}\beta_1$ and $\tan \frac{1}{2}\beta_2$ as functions of the normalized frequency x have to satisfy Foster's reactance theorem; for instance, they have to be odd rational functions of x; all poles and zeros are simple and occur at real frequencies; zeros and poles are alternating; at $x=0$ and $x=\infty$ no other values than 0 or ∞ are permitted. The degrees of denominator and numerator of each expression differ by one.

$y=\tan \frac{1}{2}(\beta_1-\beta_2)$ is a function of a less restricted character. Like $\tan \frac{1}{2}\beta_1$ and $\tan \frac{1}{2}\beta_2$ it is an odd rational function of x which is zero or infinity at zero and infinite frequency. But its zeros and poles need not alternate or occur at real frequencies, and the degree of denominator and numerator can differ widely.

This follows directly from (7) and is, because of (6b), equally valid for y and $1/E$. Dealing now with y only, since it is required that y is approximately equal to unity over a band from, say, $x=\sqrt{k}$ to $x=1/\sqrt{k}$, it is obviously not permissible to have any poles or zeros of y within this band. On the other hand, we have seen that at $x=0$ and $x=\infty$, y will be 0 or ∞, and therefore y will tend to deviate more and more from unity for very large and very small values of x. It seems plausible, therefore, to recommend (see, e.g., Hodgson[9]) that no poles or zeros should occur at real x values except at 0 and ∞, as such poles or zeros would tend to increase the deviation of y from unity; the poles or zeros at 0 and ∞ should be of the first degree. Then the degrees of numerator and denominator must differ by one. We shall see in the next section that Taylor and Tchebycheff approximations lead to expressions which are in agreement with this recommendation.

In what follows we shall assume that at $x=0$ we have $y=0$.[14] Then y will be of the form

$$y = Hx\frac{(x^2+c_1^2)(x^2+c_2^2)\cdots(x^2+c_a^2)}{(x^2+d_1^2)(x^2+d_2^2)\cdots(x^2+d_b^2)} \quad (8a)$$

where $b-a=0$ or 1, and where the constants c_1^2, $c_2^2 \cdots$ and d_1^2, $d_2^2 \cdots$ and H are real and positive. We shall denote by n the highest degree of x occurring in y, and

[14] This assumption entails no loss of generality as the only other possible choice is $y=\infty$ at $x=0$. However, in this case we would have $1/y=0$ at $x=0$, and then we could apply the results of the following discussion to $1/y$ which approximates unity as closely as y does.

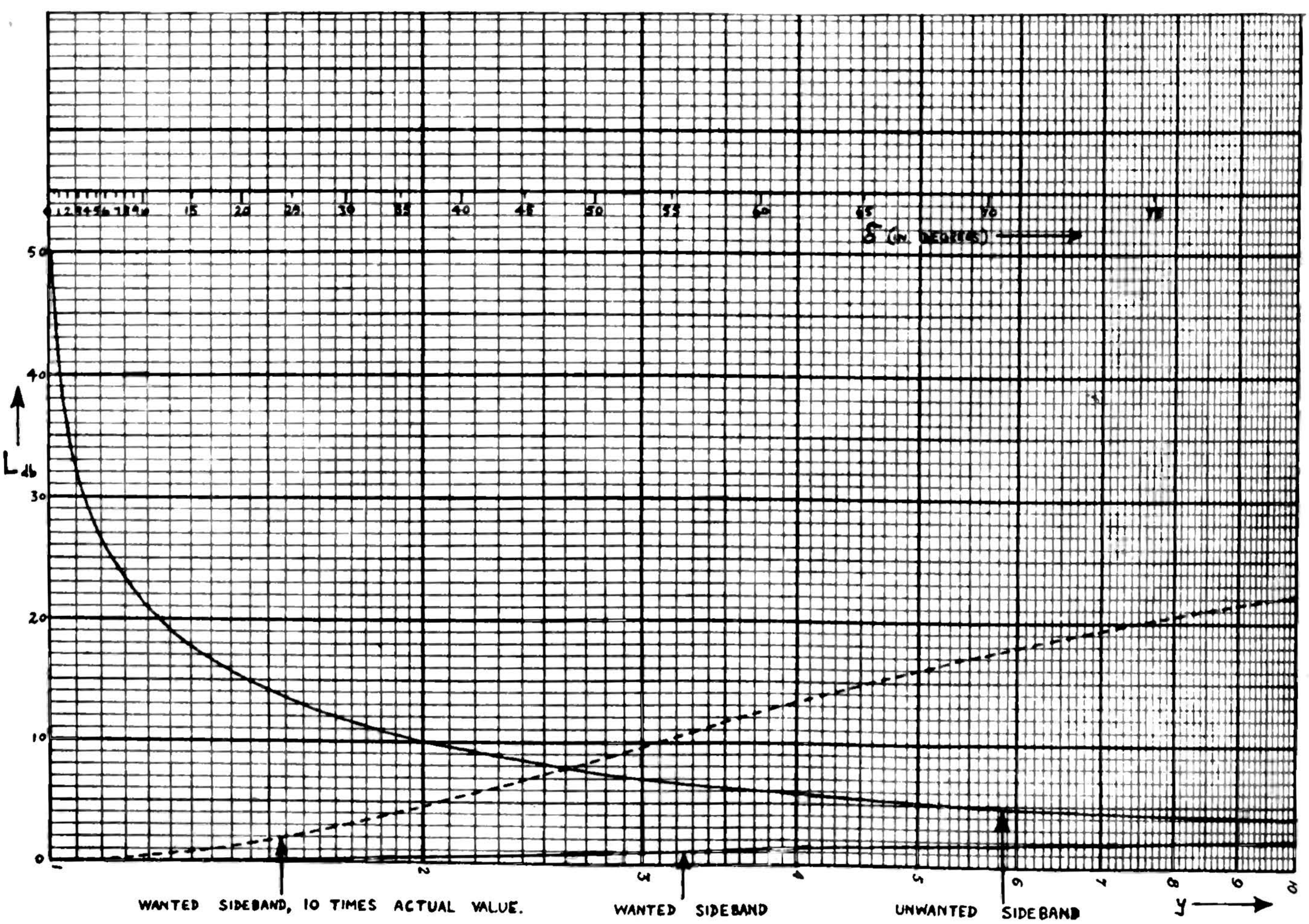


Fig. 3—Output (wanted and unwanted sideband) as function of deviation from 90° phase difference.

we shall see later that the order of approximation can be denoted by the same number n. Equation (8a) can also be written in the form

$$y = x \frac{A_0 + A_2x^2 + \cdots + A_{2a}x^{2a}}{1 + B_2x^2 + \cdots + B_{2b}x^{2b}} \tag{8b}$$

where all A's and B's are real. An important case arises when y as a function of $\log x$ is symmetrical about $x=1$, i.e., $\log x=0$. Then the transformation $x \rightarrow (1/x)$ leads to $y \rightarrow (1/y)$ if n is odd, but it leaves y unchanged if n is even. Expressions for y_n when y_n is symmetrical are listed below for n-values from 1 to 6.

$$\left.\begin{aligned}
&y_1 = x; \quad y_2 = \frac{Hx}{1+x^2}; \quad y_3 = x\frac{a+x^2}{1+ax^2};\\
&y_4 = \frac{Hx(1+x^2)}{(1+ax^2)\left(1+\frac{1}{a}x^2\right)}\\
&y_5 = x\frac{(a+x^2)(b+x^2)}{(1+ax^2)(1+bx^2)};\\
&y_6 = \frac{Hx(1+ax^2)\left(1+\frac{1}{a}x^2\right)}{(1+x^2)(1+bx^2)\left(1+\frac{1}{b}x^2\right)}
\end{aligned}\right\}. \tag{8c}$$

IV. Approximation of the Required Performance Curve

In this section we shall discuss methods for finding such values for the constants in the expressions for y ((8a), (8b), or (8c)) that y becomes a good approximation to unity in the range $x=\sqrt{k}$ to $x=1/\sqrt{k}$. If another value for y, say $y=y_0$, is required, y has to be replaced in the discussion that follows by y/y_0. We shall start with Taylor and Tchebycheff approximations, for there it is possible to go beyond a recommendation of methods of approximation to a statement of explicit formulas which give the constants in terms of k, i.e., of the range of x.

1. Taylor Approximations

A Taylor approximation of the nth order is characterized by the fact that there are n design parameters which have been so chosen that for a specified value of x, say $x=x_0$, y itself and the first $(n-1)$ differential coefficients d^ry/dx^r for $r=1, 2 \cdots (n-1)$ are the same for the required curve and the approximating curve. Thus the higher n is, the more closely the approximating curve approximates the required one.

If we assume that $x_0=1$, then the Taylor approximation of the nth order is given by

$$y_n = \tanh\left[n \tanh^{-1} x\right] \tag{9a}$$

By writing (9a) in the form

$$y_n = \frac{(1+x)^n - (1-x)^n}{(1+x)^n + (1-x)^n} \tag{9b}$$

we see that the highest degree of x occurring in y_n is n and that y_n is an odd rational function of x, symmetrical about $x=1$ against a logarithmic frequency scale.

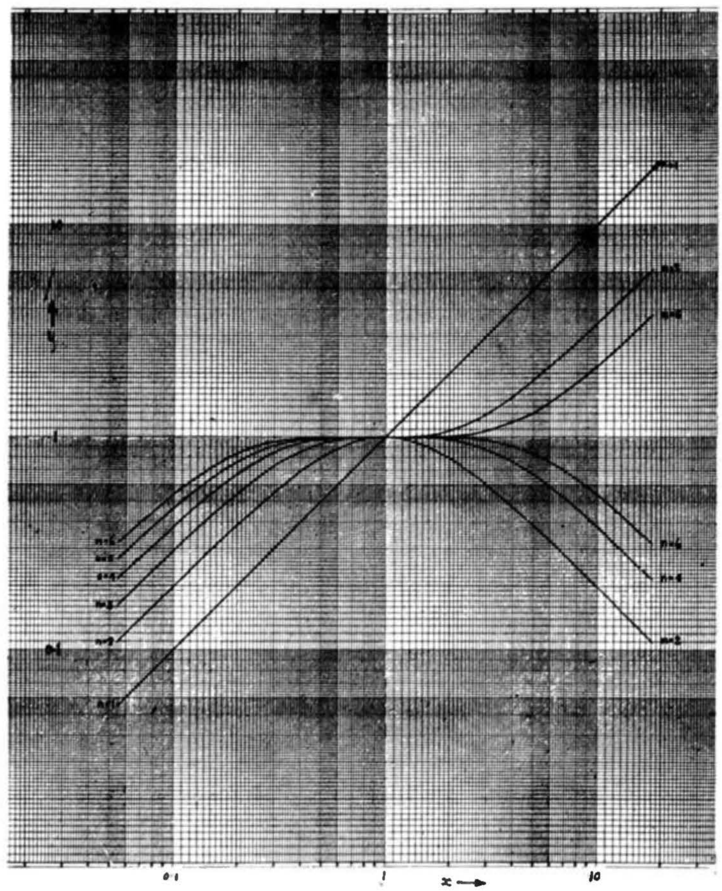


Fig. 4 (a)—Taylor approximations.

By writing it in the form

$$\frac{1-y_n}{1+y_n} = \left(\frac{1-x}{1+x}\right)^n \tag{9c}$$

we can easily prove (putting $y_n=1+\epsilon$ and $x=1+\Delta$) that the first $n-1$ differential coefficients at $x=1$ are zero, as required for a Taylor approximation to $y=1$. It may be convenient to list y_n for $n=1, 2 \cdots 6$.

$$\left.\begin{aligned}
&y_1 = x; \quad y_2 = \frac{2x}{1+x^2}; \quad y_3 = \frac{x(3+x^2)}{1+3x^2};\\
&y_4 = \frac{4x(1+x^2)}{1+6x^2+x^4}; \quad y_5 = \frac{x(5+10x^2+x^4)}{1+10x^2+5x^4};\\
&y_6 = \frac{2x(3+10x^2+3x^4)}{1+15x^2+15x^4+x^6}
\end{aligned}\right\}. \tag{9d}$$

These curves are plotted in Fig. 4(a) on log-log paper, for an x range from $x=\sqrt{k}=\sqrt{0.003}$ to $x=1/\sqrt{k}$, i.e.,

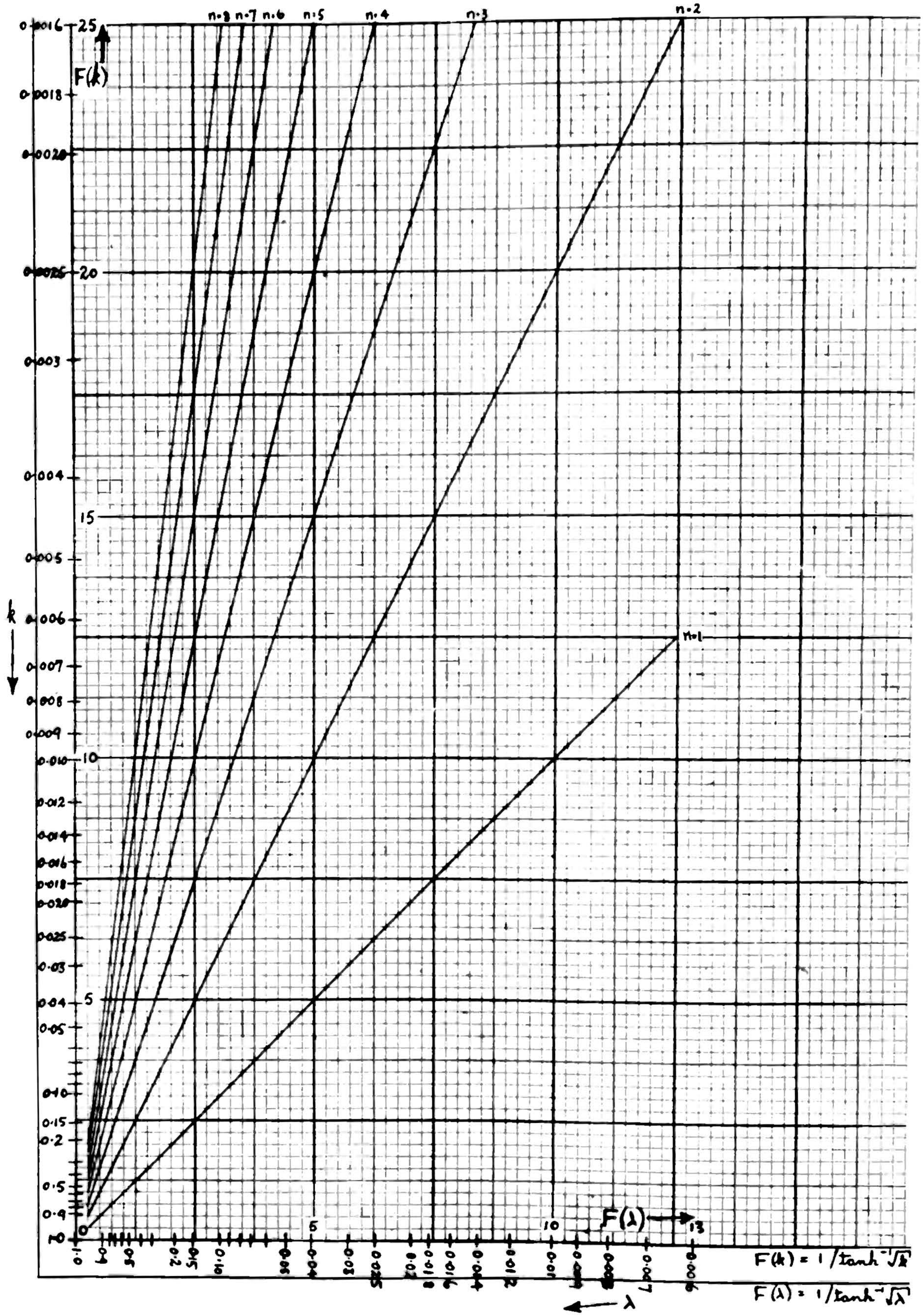


Fig. 4 (b)—Deviation of Taylor approximations from required performance.

for $k=0.003$. This corresponds, for instance, to a frequency range from 30 cps to 10 kc.

It will be see from (9b) that if x is replaced by $1/x$ y_n remains unchanged for even n values, and is replaced by $1/y_n$ for odd values of n. The deviation of $|\log y_n|$ from 0 increases with $|\log x|$. If the limits of the x range are denoted as $\sqrt{k}$ and $1/\sqrt{k}$ and the limits of the y range as $\sqrt{\lambda}$ and $1/\sqrt{\lambda}$, then λ as a function of k is given by

$$\sqrt{\lambda} = \tanh\,[n \tanh^{-1}\sqrt{k}], \tag{9e}$$

which, as far as the functional relation is concerned, is similar to (9a). If we write (9e) in the form

$$1/\tanh^{-1}\sqrt{\lambda} = \frac{1}{n}\,(1/\tanh^{-1}\sqrt{k}) \tag{9f}$$

we see that λ as a function of k can be represented as a straight line with slope $1/n$ through the origin of the co-ordinate system, for any n value, if we use functional scales defined by the $1/\tanh^{-1}\sqrt{\ }$ function for λ and k. This has been done in Fig. 4(b). In view of the functional similarity between (9f) and (9a), Fig. 4(b) also represents y_n as a function of x, in other words, Fig. 4(b) can be looked upon as showing the same curves as Fig. 4(a). It will be seen that if k is given, $|\log \lambda|$ decreases with increasing n, i.e., the range of y becomes smaller.

For synthesizing networks which have the performance described by (9a) we must find—as will be explained in Section V—the values of x at which $y = +j$. They are given by

$$x = j \tan\left[\frac{\pi}{n}\left(\frac{1}{4} + m\right)\right] \tag{9g}$$

$$\text{where}\quad m = 0, 1, 2, \cdots, (n-1).$$

2. Tchebycheff Approximations

A Tchebycheff approximation is characterized by the fact that the maximum deviation occurring is a minimum. The theory of the transformation of elliptic functions very conveniently describes odd rational functions of x, symmetrical against a logarithmic x scale about $x=1$, which over the range $x=\sqrt{k}$ to $x=1/\sqrt{k}$ approximate $y=1$, within the limits $\sqrt{\lambda}$ and $1/\sqrt{\lambda}$, in the Tchebycheff manner. As stated above, such limits for y are equivalent to limits δ_{max} and $\delta_{min} = -\delta_{max}$ for the deviation δ of the phase difference ψ from the required value 90°, and $\tan \frac{1}{2}\delta_{max} = (1-\sqrt{\lambda})/(1+\sqrt{\lambda})$. Using Cayley's[15] symbols, it can easily be shown that the Tchebycheff approximation of the nth order is given by

$$y_n/\sqrt{\lambda} = sn(u/M, \lambda) \tag{10a}$$

$$x/\sqrt{k} = sn(u, k). \tag{10b}$$

The highest degree of x occurring in the rational function defined by (10a) and (10b) is n. Cayley uses the suffix '1' for λ and M to indicate that a "second transformation" from a modulus k to a larger modulus λ is meant. However, in the following discussion it is convenient to use in many cases the suffix n to denote the order of the transformation. Therefore, in order to avoid confusion, Cayley's suffix '1' will not be used. For the purposes of this discussion it is also convenient to denote λ sometimes as k_n. In equations (10a) and (10b) u is an auxiliary variable which is defined by (10b), and y_n is defined in terms of u by (10a). k has been defined above. λ and M can be derived from k as follows: K'/K is a function of k, say $K'/K = F(k)$ known in the theory of elliptic functions (tabulated for instance by Hayashi[16]). Λ'/Λ denotes the same function of λ so that $\Lambda'/\Lambda = F(\lambda)$. λ can be obtained from k by means of the relation

$$\Lambda'/\Lambda = F(\lambda) = \frac{1}{n}K'/K = \frac{1}{n}F(k). \tag{10c}$$

It will be shown that if k is given, $|\log \lambda|$ decreases with increasing n, i.e., the range of y becomes smaller. Furthermore, K can be found as a function of k, and Λ as the same function of λ, in Hayashi's tables. Then M is given by

$$M = K/\Lambda. \tag{10d}$$

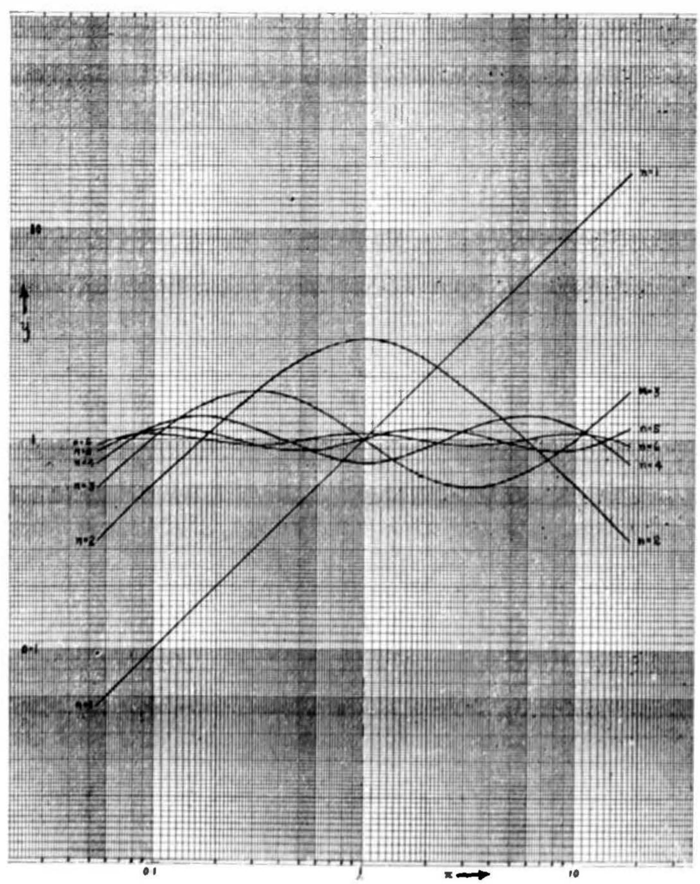


Fig. 5 (a)—Tchebycheff approximations.

[15] A. Cayley, "Elliptic Functions," 2nd ed., George Bell & Sons, London; 1895.

[16] K. Hayashi, "Tafeln der Besselschen, Theta, Kugel- und anderer Funktionen," Berlin; 1930.

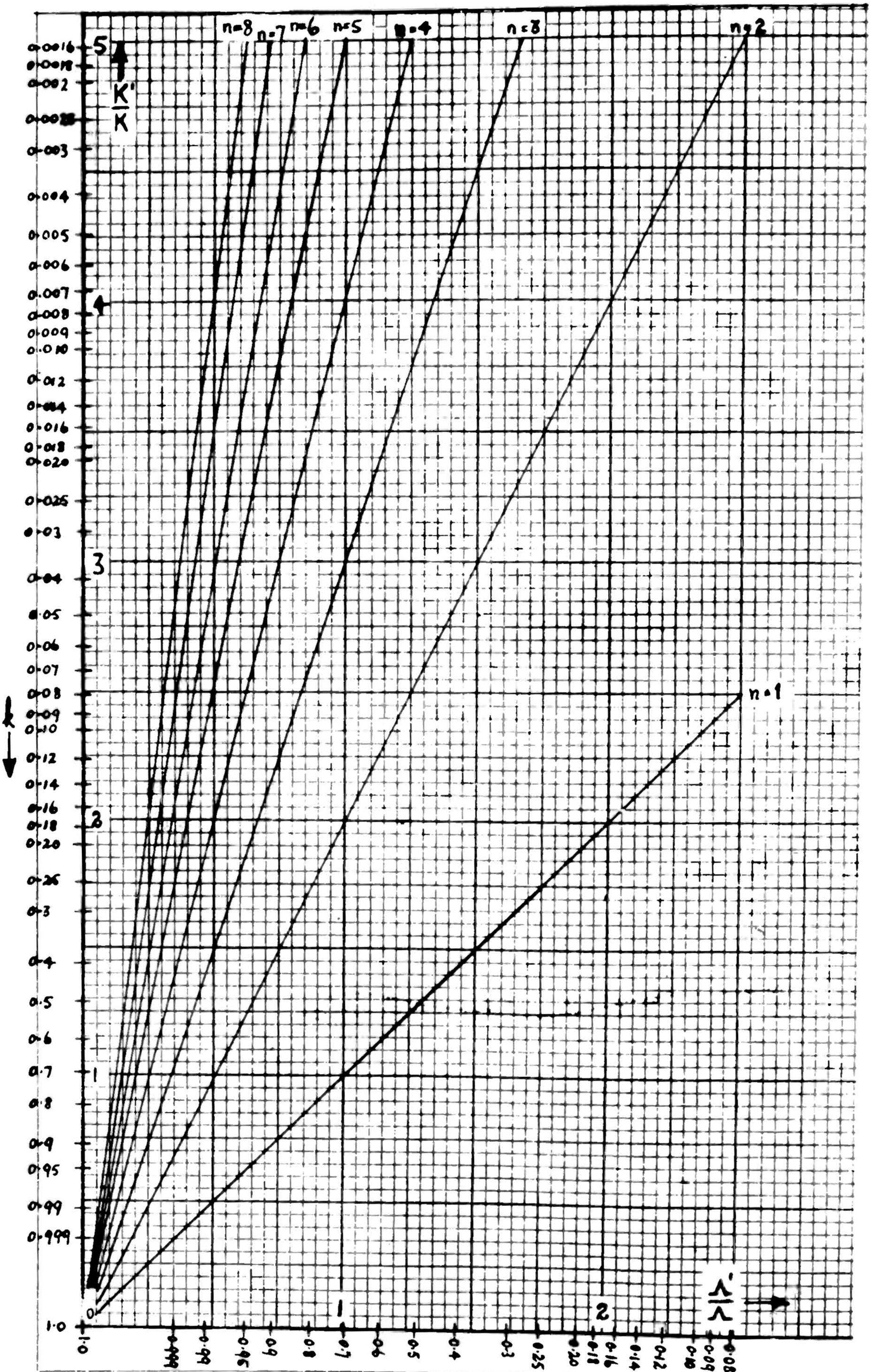


Fig. 5 (b)—Deviation of Tchebycheff approximations from required performance.

TABLE Ia

TCHEBYCHEFF APPROXIMATIONS; y AND u AS FUNCTIONS OF x.

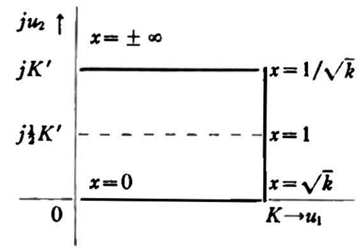
$$\frac{y_n}{\sqrt{\lambda}} = sn\left(\frac{u}{M_n}, \lambda\right) \tag{10a}$$

$$\frac{x}{\sqrt{k}} = sn(u, k) \tag{10b}$$

$$M_n = \frac{K}{\Lambda} = \frac{K'}{n\Lambda'}$$

x as a function of $u=u_1+ju_2$ is described by the following table and diagram

x	$0 \cdots \sqrt{k}$	$\sqrt{k} \cdots 1$	$1 \cdots 1/\sqrt{k}$	$1/\sqrt{k} \cdots \infty$
u_1	$0 \cdots K$	K	K	$K \cdots 0$
u_2	0	$0 \cdots \frac{1}{2}K'$	$\frac{1}{2}K' \cdots K'$	K'



Best form of (10b)

for $x = 0 \cdots \sqrt{k}$	$x/\sqrt{k} = sn(u_1, k)$,	$u_2 = 0$	
for $x = \sqrt{k} \cdots 1/\sqrt{k}$	$x/\sqrt{k} = 1/dn(u_2, k')$,	$u_1 = K$,	$k' = \sqrt{1-k^2}$
for $x = 1/\sqrt{k} \cdots \infty$	$x/\sqrt{k} = 1/[k\, sn(u_1, k)]$,	$u_2 = K'$	

Best form of expression giving y as a function of x if $x = \sqrt{k} \cdots 1/\sqrt{k}$

$$\frac{y}{\sqrt{\lambda}} = \frac{1}{dn\left[\frac{1}{M_n} dn^{-1}\left(\frac{\sqrt{k}}{x}, k'\right), \lambda'\right]}, \qquad \lambda' = \sqrt{1-\lambda^2}$$

Thus $\lambda = k_n$ can be obtained for any k and n, and, k being specified, n can be so chosen as to make log $\sqrt{\lambda}$, which denotes the maximum deviation of $|\log y|$ from 0, as small as required. λ as a function of k and n is represented in Fig. 5(b). Since (10c) is of the same form as (9f) it is again possible to draw the λ curves as straight lines with slope $1/n$ if linear scales for K'/K and Λ'/Λ are used. It should be noted that in the case of Tchebycheff approximations the curves relating λ to k (Fig. 5b) do not at the same time relate y_n to x. It will be seen that for any given k- and n-values the values of λ obtained from Fig. 5(b) i.e., for Tchebycheff approximations are much nearer to 1 than those obtained from Fig. 4(b), i.e., for Taylor approximations.

y_n as a function of x can be evaluated directly from (10a) and (10b) by means of tables of elliptic functions, e.g., Milne-Thomson's[17] tables together with Hayashi's tables. For this purpose, equations (10a) and (10b) can be modified as shown in Table I(a). The Tchebycheff approximations for $n=1, 2 \cdots 6$ for an x-range from $\sqrt{k}$ to $1/\sqrt{k}$ where $k=0.003$ are shown diagrammatically in Fig. 5(a). For these diagrams the formulas given in Table I(a) have not been used. Only the x values at which maxima or minima of y occur and those at which $y=1$ have been evaluated numerically, and the curves have been so drawn as to go through these points. However, for $n=4$ (see Section VI) a numerical check for a great number of points has shown very good agreement with the drawn curve. The curves have an oscillatory behavior, all maxima and minima occur at the y-values $1/\sqrt{\lambda}$ and $\sqrt{\lambda}$ respectively, the value of λ depending on the n value and the k value under consideration. For even n values there are $\frac{1}{2}n$ maxima, $\frac{1}{2}(n-2)$ minima, two intersections with the line $y=\sqrt{\lambda}$ and n intersections with the line $y=1$. For odd n values there are $\frac{1}{2}(n-1)$ minima and $\frac{1}{2}(n-1)$ maxima, one intersection with the line $y=\sqrt{\lambda}$, one intersection with the line $y=1/\sqrt{\lambda}$ and n intersections with the line $y=1$.[18]

In order to be able to plot these characteristic points of y we must know the values of x at which $y_n=\sqrt{\lambda}$, 1 and $1/\sqrt{\lambda}$. On the other hand, in order to be able to write y_n as a rational function of x in the form of equation (8c) we must know the values of x at which $y_n=0$ and $y_n=\infty$ and the value of H in the case of even n values (see below). Lastly, in order to synthesize a net-

[17] L. M. Milne-Thomson, "Die elliptischen Funktionen von Jacobi," Julius Springer, Berlin; 1931.

[18] It is interesting to note that in the case of filter design a "first transformation" from a modulus k to a smaller modulus λ has to be used. This transformation, though similar in many respects to the one used in our problem, differs from the one defined by equations (10a) and (10b) in so far as it leads to a rational function of x only for odd n values. For even n values, in order to obtain y as a rational function of x, x has, as Darlington has shown, to be defined by a more complicated relation between x and u (see reference in footnote 27).

TABLE Ib

TCHEBYCHEFF APPROXIMATIONS; x AS A FUNCTION OF y.

$$\frac{y_n}{\sqrt{\lambda}} = sn\left(\frac{u}{M_n}, \lambda\right) \qquad \frac{x}{\sqrt{k}} = sn(u, k)$$

$$x = \sqrt{k}\, sn \left\{ \left[M_n\, sn^{-1}\left(\frac{y_n}{\sqrt{\lambda}}, \lambda\right) + j2q \frac{K'}{n} \right], k \right\}, \qquad q = 0, 1, 2, 3, \cdots, (n-1).$$

Let $M_n sn^{-1}\left(\frac{y_n}{\sqrt{\lambda}}, \lambda\right) = T$, then

y_n	T	x
0	0	$j\sqrt{k}\, sc\left[2q \frac{K'}{n}, k'\right]$
j	$j \frac{1}{2} \frac{K'}{n}$	$j\sqrt{k}\, sc\left[\left(\frac{1}{2} + 2q\right) \frac{K'}{n}, k'\right]$
∞	$j \frac{K'}{n}$	$j\sqrt{k}\, sc\left[(1 + 2q) \frac{K'}{n}, k'\right]$
$-j$	$j \frac{3}{2} \frac{K'}{n}$	$j\sqrt{k}\, sc\left[\left(\frac{3}{2} + 2q\right) \frac{K'}{n}, k'\right]$
$\sqrt{\lambda}$	K	$\sqrt{k}\, nd\left[2q \frac{K'}{n}, k'\right]$
1	$K + j\frac{K'}{n}\left(\frac{1}{2} \text{ or } \frac{3}{2}\right)$	$\sqrt{k}\, nd\left[\left(\frac{1}{2} + q\right) \frac{K'}{n}, k'\right]$
$1/\sqrt{\lambda}$	$K + j\frac{K'}{n}$	$\sqrt{k}\, nd\left[(1 + 2q) \frac{K'}{n}, k'\right]$

$$sc(u, k) = \frac{sn(u, k)}{cn(u, k)}, \qquad nd(u, k) = \frac{1}{dn(u, k)}$$

This table is given in greater detail in Tables II and III.

work having a performance curve in accordance with y, we have to find the values of x at which $y = +j$. All these values of x can be found from (10a) and (10b) by first inverting (10a) to find u as a function of y and then substituting this expression for u in (10b). However, to simplify the engineering application of Tchebycheff approximations, the x values at which y becomes $0, j, \infty, -j$ and $\sqrt{\lambda}, 1, 1/\sqrt{\lambda}$, are listed in Tables I, II, and III. The expressions tabulated are so regular in form that it is easy, if required, to extend by analogy the Table to any n value. The value H mentioned above is given by

$$H = \sqrt{\lambda/k}/M_n. \tag{11}$$

When dealing with Tchebycheff approximations it is often convenient to make use of the "index law" which is valid for these approximations. Let $y_n(x, k)$ denote the nth order approximation to $y_n = 1$ over the x range $\sqrt{k}$ to $1/\sqrt{k}$, and let k_n denote the range of variation of $y_n(x, k)$, i.e., y_n varies between $\sqrt{k_n}$ and $1/\sqrt{k_n}$. Furthermore, let $y_m(y_n, k_n)$ denote the mth order approximation to $y_m = 1$ over the y_n range $\sqrt{k_n}$ to $1/\sqrt{k_n}$, and let $(k_n)_m$ denote the range of variation of y_m, i.e., y_m varies between $\sqrt{(k_n)_m}$ and $1/\sqrt{(k_n)_m}$. Then $y_m(y_n, k_n)$ considered as a function of x when x varies from $\sqrt{k}$ to $1/\sqrt{k}$ is identical with $y_p(x, k)$, the pth order approximation to $y_p = 1$, over the x range $\sqrt{k}$ to $1/\sqrt{k}$, if $p = mn$. This can be expressed formally by

$$\left.\begin{aligned} y_m(y_n, k_n) &= y_p(x, k), & p &= mn \\ (k_n)_m &= k_p, & p &= mn \end{aligned}\right\}. \tag{12}$$

By virtue of the index law we can, if we have explored the case of $n = 2$, apply all results obtained to $n = 4 = 2 \times 2$ and $n = 8 = 2 \times 4$. If we have explored $n = 2$ and $n = 3$, we can combine the results to obtain the cases $n = 6 = 2 \times 3$ and $n = 9 = 3 \times 3$. A generalizing interpretation of the index law will be given in the Appendix.

3. Alternative Theory of Tchebycheff Approximations

So far, the theory of Tchebycheff approximations has been discussed in terms of elliptic functions. This leads to the most concise and general type of expressions. At the same time it must be realized that many engineers are unfamiliar with elliptic functions and that it is sometimes difficult to obtain good tables of elliptic functions. It is therefore important to note that it is possible to formulate the approximations purely algebraically, without the use of elliptic functions. In practice, a combination of the two methods of attack, appropriate to the particular case under consideration, is sometimes the best choice.

The algebraic theory for $n=2, 4, 8$ is very simple indeed. Starting with $n=2$, $y_2=d_2x/(1+x^2)$ leads to the following relations:

for $x=\sqrt{k}$ and $x=1/\sqrt{k}$, $y_2=y_{2\min}=d_2\sqrt{k}/(1+k)$;

and for $x=1$, $y_2=y_{2\max}=\frac{1}{2}d_2$.

The condition $y_{2\min}y_{2\max}=1$ leads to $k_2=2\sqrt{k}/(1+k)$ and $d_2=2/\sqrt{k_2}$. With this value for d_2, y_2 is the Tchebycheff approximation of the second order for the range k. The cases $n=4$ and $n=8$ can be discussed by applying the index law. The results are tabulated in Table IV.

TABLE II

TCHEBYCHEFF APPROXIMATIONS; x FOR $y=0, +j, \infty, -j$.

$n=1$

x	y_1
0	0
$+j$	$+j$
$\pm\infty$	$\pm\infty$
$-j$	$-j$
0	0

$n=2$

x	y_2
$j\sqrt{k}\,sc(0) \quad =0$	0
$j\sqrt{k}\,sc\left(\frac{K'}{4}, k'\right)$	$+j$
$j\sqrt{k}\,sc\left(\frac{K'}{2}, k'\right)=+j$	$\pm\infty$
$j\sqrt{k}\,sc\left(\frac{3}{4}K', k'\right)$	$-j$
$j\sqrt{k}\,sc(K', k') \quad =\infty$	0
$j\sqrt{k}\,sc\left(\frac{5}{4}K', k'\right)$	$+j$
$j\sqrt{k}\,sc\left(\frac{3}{2}K', k'\right)=-j$	$\pm\infty$
$j\sqrt{k}\,sc\left(\frac{7}{4}K', k'\right)$	$-j$
$j\sqrt{k}\,sc(0) \quad =0$	0

$$sc(u, k)=\frac{sn(u, k)}{cn(u, k)}=sc(u+2K, k)$$
$$sc(2K-u, k)=sc(-u, k)=-sc(u, k)$$

$n=3$

x	y_3
$j\sqrt{k}\,sc(0) \quad =0$	0
$j\sqrt{k}\,sc\left(\frac{K'}{6}, k'\right)$	$+j$
$j\sqrt{k}\,sc\left(\frac{K'}{3}, k'\right)$	$\pm\infty$
$j\sqrt{k}\,sc\left(\frac{K'}{2}, k'\right)=+j$	$-j$
$j\sqrt{k}\,sc\left(\frac{2}{3}K', k'\right)$	0
$j\sqrt{k}\,sc\left(\frac{5}{6}K', k'\right)$	$+j$
$j\sqrt{k}\,sc(K', k') \quad =\infty$	$\pm\infty$
$j\sqrt{k}\,sc\left(\frac{7}{6}K', k'\right)$	$-j$
$j\sqrt{k}\,sc\left(\frac{4}{3}K', k'\right)$	0
$j\sqrt{k}\,sc\left(\frac{3}{2}K', k'\right)=-j$	$+j$
$j\sqrt{k}\,sc\left(\frac{5}{3}K', k'\right)$	$\pm\infty$
$j\sqrt{k}\,sc\left(\frac{11}{6}K', k'\right)$	$-j$
$j\sqrt{k}\,sc(0) \quad =0$	0

In the General Case

x	y_n
$j\sqrt{k}\,sc(0) \quad =0$	0
$j\sqrt{k}\,sc\left(\frac{1}{2n}K', k'\right)$	$+j$
$j\sqrt{k}\,sc\left(\frac{2}{2n}K', k'\right)$	$\pm\infty$
$j\sqrt{k}\,sc\left(\frac{3}{2n}K', k'\right)$	$-j$
⋮	⋮

TABLE III

Tchebycheff Approximations; x for $y=\sqrt{\lambda}$, 1, $1/\sqrt{\lambda}$

$n=1$	
x	y_1
$\sqrt{k}$	min
1	1
$1/\sqrt{k}$	max

$n=2$		
x		y_2
$x_1=\sqrt{k}\,nd(0)$	$=\sqrt{k}$	min
$x_2=\sqrt{k}\,nd\left(\frac{1}{4}K',k'\right)$		1
$x_3=\sqrt{k}\,nd\left(\frac{2}{4}K',k'\right)$	$=1$	max
$1/x_2=\sqrt{k}\,nd\left(\frac{3}{4}K',k'\right)$		1
$1/x_1=\sqrt{k}\,nd\left(\frac{4}{4}K',k'\right)$	$=1/\sqrt{k}$	min

$n=3$		
x		y_3
$x_1=\sqrt{k}\,nd(0)$	$=\sqrt{k}$	min
$x_2=\sqrt{k}\,nd\left(\frac{1}{6}K',k'\right)$		1
$x_3=\sqrt{k}\,nd\left(\frac{2}{6}K',k'\right)$		max
$x_4=\sqrt{k}\,nd\left(\frac{3}{6}K',k'\right)$	$=1$	1
$1/x_3=\sqrt{k}\,nd\left(\frac{4}{6}K',k'\right)$		min
$1/x_2=\sqrt{k}\,nd\left(\frac{5}{6}K',k'\right)$		1
$1/x_1=\sqrt{k}\,nd\left(\frac{6}{6}K',k'\right)$	$=1/\sqrt{k}$	max

In the General Case		
x		y_n
$x_1=\sqrt{k}\,nd(0)$	$=\sqrt{k}$	min
$x_2=\sqrt{k}\,nd\left(\frac{1}{2n}K',k'\right)$		1
$x_3=\sqrt{k}\,nd\left(\frac{2}{2n}K',k'\right)$		max
$x_4=\sqrt{k}\,nd\left(\frac{3}{2n}K',k'\right)$		1
⋮		⋮

$$nd(u,k)=1/dn(u,k)=nd(-u,k)$$
$$nd(u+2K,k)=nd(u,k)$$
$$nd(K'\pm u,k')=1/[k\,nd(u,k')]$$

For $n=3$ the derivation of an expression for y_3 is less simple. We start with an expression of the form $y_3=Hx(A_0+x^2)/(B_0+x^2)$, which, in order to be symmetrical about $x=1$, simplifies to $y=x(a+x^2)/(1+ax^2)$. Then we have to determine a so that, for a given range k, y behaves in a Tchebycheff manner. y is required to be equal to $\sqrt{\lambda}$ if $x=\sqrt{k}$; furthermore, y has also the value $\sqrt{\lambda}$ as a minimum value at an unknown x value, say $x=b$. Therefore

$$y-\sqrt{\lambda}=[x(a+x^2)-\sqrt{\lambda}(1+ax^2)]/(1+ax^2)$$

must be equal to $(x-\sqrt{k})(x-b)^2/(1+ax^2)$. Comparing coefficients we obtain three equations:

$$a\sqrt{\lambda}=\sqrt{k}+2b,\quad a=b^2+2b\sqrt{k}\quad\text{and}\quad\sqrt{\lambda}=b^2\sqrt{k}.$$

If we introduce α, a term used by Cayley, by means of

$$b=\sqrt{k}/\alpha$$

we find

$$a=k/\alpha^2+2k/\alpha \tag{13a}$$

and

$$k^2=\alpha^3\frac{2+\alpha}{1+2\alpha},\qquad \lambda^2=\alpha\left(\frac{2+\alpha}{1+2\alpha}\right)^3. \tag{13b}$$

Equations (13a) and (13b) determine α, a and λ in terms of k (λ can also be determined by means of tables of elliptic functions, or by means of a graph like that in Fig. 5(b)). Then we can find $b^2=\sqrt{\lambda/k}$. Thus we know y as a function of x and the following details:

$y=\sqrt{\lambda}$ at $x=\sqrt{k}$ and $x=b$

$y=1/\sqrt{\lambda}$ at $x=1/\sqrt{k}$ and $x=1/b$

$y=1$ at $x=1$ and $x=\frac{1}{2}(a-1)\pm\sqrt{\frac{1}{4}(a-1)^2-1}$.

TABLE IV

TCHEBYCHEFF APPROXIMATIONS FOR $n=2, 4, 8$; ALGEBRAIC RELATIONS

$x=[F(d_2/y_2)]^{\pm 1}$	$y_2=\frac{d_2x}{1+x^2}=[F(d_4/y_4)]^{\pm 1}$	$y_4=\frac{d_4y_2}{1+y_2^2}=[F(d_8/y_8)]^{\pm 1}$	$y_8=\frac{d_8y_4}{1+y_4^2}$
$\sqrt{k}$	min	min	min
$F\{d_2/F[d_4/F(d_8)]\}$	$F[d_4/F(d_8)]$	$F(d_8)$	1
$F[d_2/F(d_4)]$	$F(d_4)$	1	max
$F\{d_2/F[d_4F(d_8)]\}$	$F[d_4F(d_8)]$	$1/F(d_8)$	1
$F(d_2)$	1	max	min
$F\{d_2F[d_4F(d_8)]\}$	$1/F[d_4F(d_8)]$	$1/F(d_8)$	1
$F[d_2F(d_4)]$	$1/F(d_4)$	1	max
$F\{d_2F[d_4/F(d_8)]\}$	$1/F[d_4/F(d_8)]$	$F(d_8)$	1
1	max	min	min
$1/F\{d_2F[d_4/F(d_8)]\}$	$1/F[d_4/F(d_8)]$	$F(d_8)$	1
$1/F[d_2F(d_4)]$	$1/F(d_4)$	1	max
$1/F\{d_2F[d_4F(d_8)]\}$	$1/F[d_4F(d_8)]$	$1/F(d_8)$	1
$1/F(d_2)$	1	max	min
$1/F\{d_2/F[d_4F(d_8)]\}$	$F[d_4F(d_8)]$	$1/F(d_8)$	1
$1/F[d_2/F(d_4)]$	$F(d_4)$	1	max
$1/F\{d_2/F[d_4/F(d_8)]\}$	$F[d_4/F(d_8)]$	$F(d_8)$	1
$1/\sqrt{k}$	min	min	min
	min: $y_2=\sqrt{k_2}$, max: $y_2=\frac{1}{\sqrt{k_2}}$	min: $y_4=\sqrt{k_4}$, max: $y_4=\frac{1}{\sqrt{k_4}}$	min: $y_8=\sqrt{k_8}$, max: $y_8=\frac{1}{\sqrt{k_8}}$

$$y_2=+j \quad \text{if} \quad x=[F(d_2/j)]^{\pm 1}$$
$$y_4=+j \quad \text{if} \quad y_2=[F(d_4/j)]^{\pm 1}, \quad x=[F(d_2/y_2)]^{\pm 1}$$
$$y_8=+j \quad \text{if} \quad y_4=[F(d_8/j)]^{\pm 1}, \quad y_2=[F(d_4/y_4)]^{\pm 1}, \quad x=[F(d_2/y_2)]^{\pm 1}$$

$$k_2=\frac{2\sqrt{k}}{1+k}, \qquad k_4=\frac{2\sqrt{k_2}}{1+k_2}, \qquad k_8=\frac{2\sqrt{k_4}}{1+k_4}$$
$$d_2=2/\sqrt{k_2}, \qquad d_4=2/\sqrt{k_4}, \qquad d_8=2/\sqrt{k_8}$$
$$F(z)=\tfrac{1}{2}z-\sqrt{(\tfrac{1}{2}z)^2-1} \qquad 1/F(z)=\tfrac{1}{2}z+\sqrt{(\tfrac{1}{2}z)^2-1}$$

If we replace in the discussion of the case $n=3$ the independent variable x by the second-order approximation y_2 and k by k_2, we obtain a sixth-order approximation, and by repeating this process we obtain a twelfth-order approximation. If in the discussion of the case $n=3$ we replace x by y_8 and k by k_8 we obtain the ninth-order approximation. For the prime numbers $n=5, 7, 11, \cdots$ etc., however, the algebraic theory becomes progressively more difficult (see Cayley).

4. Approximation by Other Methods

In most cases the requirements concerning the performance of phase splitting networks can probably be satisfied by means of Tchebycheff or Taylor approximations which have been discussed in the preceding section. Sometimes, however, requirements may be stipulated for which these types of approximations are not the best possible solutions. Then other types of approximations have to be obtained.

If it is required to have an exact phase difference of 90° at n specified values of x, then the parameters of the function y which satisfies these requirements can be obtained by solving n linear simultaneous equations. Zobel[19] has discussed this method in great detail with reference to the design of attenuation equalizers and phase shift networks. The application to the design of phase splitting networks does not raise any new problems.

Zobel recommends the use of this method not only in cases where the performance at a number of points is specified, but also where a good approximation over a whole range of x values is required. However, in such cases Zobel's method often leads to disappointments (see e.g., comments by Saraga[20] and Baum[21]) and graphical methods of curve fitting are to be preferred.

A survey of graphical curve fitting methods shows that they can conveniently be classified as curve summation or curve shifting and shaping methods (see Saraga[20]). It is usually necessary to transform the coordinate system in which the required performance curve and its tolerance band are specified in order to make the application of these graphical methods possible. In the summation method, a curve which fits the tolerance band is obtained by adding a number of standard curves in different positions. From these positions the parameters of the approximating curve can be obtained (for examples, see Laurent,[22] Rumpelt,[23] Saraga,[20] Scowen,[24] Baum[21]). In the shifting and shaping method which can be used for a limited number (not more than 4 to 5) of parameters only, one single standard curve is shifted and shaped by scale changes and shearing until it fits the required tolerance band (see Pyrah,[25] Truscott,[26] Saraga[20]). The application of these methods to the specific problems of phase splitting networks will not be discussed here.

V. Network Synthesis

At this stage we shall assume that in one way or another a suitable performance function $y(x)$ has been determined. The next step is the determination of two phase shift networks which will produce this function y. The problem to be solved is to find X_1/R_0 and X_2/R_0 when

$$y = \frac{(X_1/R_0) - (X_2/R_0)}{1 + (X_1/R_0)(X_2/R_0)}$$

is known. As this problem occurs also in the design of symmetrical filters where (see equation (6b))

$$\frac{1 + (X_A/R_0)(X_B/R_0)}{(X_A/R_0) - (X_B/R_0)}$$

is given and X_A/R_0 and X_B/R_0 have to be found as physically possible reactances, we can apply its solution to our problem. Darlington[27] gives the following instructions for determining the reactances (modified here in accordance with the symbols used in this paper):—Write y in the form $y=xB'/P$ where B' and P are polynomials in x^2. Then express $P+pB'$ in the form $(P_1+pB_1)(P_2-pB_2)$ where P_1, B_1, P_2, B_2, are even polynomials in $p=jx$, such that the roots of $P_1+pB_1=0$ are the roots of $P+pB'=0$ (i.e. $y=+j$) which have negative real parts. Then $jX_1/R_0=pB_1/P_1$ and $jX_2/R_0=pB_2/P_2$.

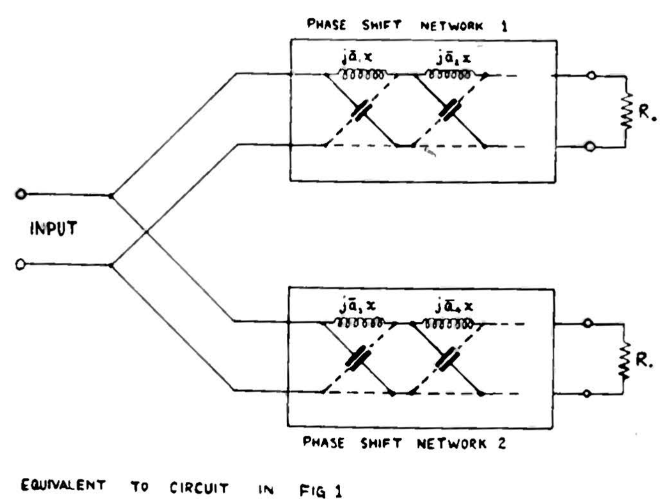


Fig. 6—Basic phase-splitting circuit, decomposed into elementary phase shift sections.

[19] O. J. Zobel, "Distortion correction in electrical circuits with constant resistance recurrent networks," *Bell Sys. Tech. Jour.*, vol. 7, pp. 438–534; July, 1928.
[20] W. Saraga, "Attenuation and phase shift equalisers," *Wireless Eng.*, vol. 20, pp. 163–181; April, 1943.
[21] R. F. Baum, "A contribution to the approximation problem," Proc. I.R.E., vol. 36, pp. 863–869; July, 1948.
[22] T. Laurent, "New principles for practical computation of filter attenuation by means of frequency transformation," *Ericsson Technics*, vol. 3, pp. 57–72; 1939.
[23] E. Rumpelt, "Schablonenverfahren fuer den Entwurf elektrischer Wellenfilter auf der Grundlage der Wellenparameter," *Telegraphen Fernsprech. Funk und Fernseh-und Technik*, vol. 31, pp. 203–210; August, 1942.
[24] F. Scowen, "Electric Wave Filters," Chapman & Hall Ltd., London, pp. 72–74; 1945.
[25] F. Pyrah, "Constant impedance equalisers: Simplified method of design and standardisation," *British P.O. Elec. Eng.'s Jour.*, vol. 92, pp. 204–211; October, 1939.
[26] D. N. Truscott, "Logarithmic charts and circuit performance," *Electronic Eng.*, vol. 14, pp. 745–748; May, 1942.
[27] S. Darlington, "Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics," *Jour. Math. Phys.*, vol. 13, pp. 257–353; September, 1939.

Darlington's instructions, given without explicit proof, can be modified in a way which includes the proof. For this purpose we consider those values of x at which $y=\tan \frac{1}{2}(\beta_1-\beta_2)=j$; at these values, say $x=x_j$, the phase-shift difference $\frac{1}{2}(\beta_1-\beta_2)$ approaches $+j\infty$ which must be due to $\frac{1}{2}\beta_1$ tending towards $+j\infty$ or $\frac{1}{2}\beta_2$ tending towards $-j\infty$. Let us now assume that the two basic phase shift networks with series reactances X_1 and X_2 (see Fig. 1) consist of "elementary" phase shift sections in tandem, each section being characterized by its phase shift β and its normalized series arm reactance $\bar{a}x=\tan \frac{1}{2}\beta$ or its normalized series arm inductance $\bar{a}$ (see Fig. 6).[28] Then at each x_j one of these elementary phase angles $\frac{1}{2}\beta$ must tend to $+j\infty$ and $\tan \frac{1}{2}\beta=j$, if β is a constituent of β_1, or to $-j\infty$ and $\tan \frac{1}{2}\beta=-j$, if β is a constituent of β_2. Since $\tan \frac{1}{2}\beta=\bar{a}x$, we find $\bar{a}=\pm j/x_j$. We take that sign which makes $\bar{a}$, the normalized inductance, positive. If, in order to obtain a positive $\bar{a}$ we have to take the positive sign, the corresponding β is a constituent of β_1, whereas in the other case we obtain a constituent of β_2. In this way we not only find X_1 and X_2, but also, at the same time, the constituent elementary sections forming the two basic phase shift networks. It can be shown that forming the expressions for X_1 and X_2 from the inductances of the elementary phase shift sections in accordance with the addition theorem of the tan-function leads to the expressions given by Darlington.

It will be seen that n elementary sections lead to an expression for y in which the highest degree of x is n, and vice versa. Thus the number of network elements increases with the highest degree of x occurring.

VI. Two Practical Design Examples

It is felt that in selecting practical examples for discussion in this article it is best to take very simple ones, as then the method of obtaining the networks can be shown most clearly. As a first example we shall discuss a case in which a Taylor approximation is required, and we select a simple case, namely $n=3$. Then the best approximation is given by

$$y_3=\frac{3x+x^3}{1+3x^2} \tag{14}$$

(see equations (9d)). Since n is odd, the number of sections of the two phase shifting networks must differ by one. Let us assume that the network with X_1, consists of two sections, say with series arm inductances $\bar{a}_1$ and $\bar{a}_2$, respectively. Then the network with X_2 has one single section, say with series arm inductance $\bar{a}_3$. Thus we obtain

$$\tan \tfrac{1}{2}\beta_1=\frac{(\bar{a}_1+\bar{a}_2)x}{1-\bar{a}_1\bar{a}_2x^2}, \quad \tan \tfrac{1}{2}\beta_2=\bar{a}_3x$$

and

$$y=\tan \tfrac{1}{2}(\beta_1-\beta_2)=\frac{(\bar{a}_1+\bar{a}_2-\bar{a}_3)x+\bar{a}_1\bar{a}_2\bar{a}_3x^3}{1+(\bar{a}_2\bar{a}_3+\bar{a}_3\bar{a}_1-\bar{a}_1\bar{a}_2)x^2}. \tag{15}$$

In this simple case we can obtain $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ by comparing coefficients in (14) and (15). Then $\bar{a}_1+\bar{a}_2-\bar{a}_3=3$;

$$\bar{a}_1\bar{a}_2\bar{a}_3=1; \qquad \bar{a}_2\bar{a}_3+\bar{a}_3\bar{a}_1-\bar{a}_1\bar{a}_2=3.$$

By substituting we obtain a cubic equation for $\bar{a}_3$ with one positive root: $\bar{a}_3=+1$. Then $\bar{a}_1=2+\sqrt{3}$, $\bar{a}_2=2-\sqrt{3}$ and $X_1/R_0=4x/(1-x^2)$, $X_2/R_0=x$. X_1 and X_2 can be interchanged. In a more complicated case we would solve the equation $y=+j$ and would obtain the three roots $x=-j$, $x=+j(2+\sqrt{3})$ and $x=+j(2-\sqrt{3})$ either by means of equation (9g) or algebraically. In view of the signs of the roots, the first one must correspond to X_2 and the other two must correspond to X_1. Thus we obtain $\bar{a}_1=2+\sqrt{3}$, $\bar{a}_2=2-\sqrt{3}$ and $\bar{a}_3=+1$ as before.

As a second example we shall discuss a case in which a Tchebycheff approximation is required. We shall take $n=4$ so that we can use an algebraic method as well as the transformation of elliptic functions for obtaining the network elements. The specified x range is assumed to be from $x=\sqrt{k}$ to $x=1/\sqrt{k}$ where $k=0.003$. This corresponds to a frequency range from 30 cps to 10 kc. Then by means of Hayashi's tables $k_4=\lambda$ is found to be 0.5959. The y curve is shown in Fig. 5(a). Table II gives the expressions for the four x values at which $y=+j$. Then using Milne-Thomson's tables, we find $x=+j2.469$, $x=-j/2.469$, $x=+j0.05618$, $x=-j/0.05618$. Then for one phase-shift network $\bar{a}_1=2.469$, $\bar{a}_2=0.05618$ and for the other network $\bar{a}_3=1/\bar{a}_2=17.80$, $\bar{a}_4=1/\bar{a}_1=0.4049$. From these values of $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, $\bar{a}_4$ we find

$$X_1/R_0=2.526x/(1-0.1387x^2),$$
$$X_2/R_0=18.20x/(1-7.208x^2)$$

and $y_4=15.68x(1+x^2)/[1+38.62x^2+x^4]$. X_1 and X_2 can be interchanged. Applying the algebraic theory we obtain from Table IV

$$d_2=6.052, \qquad d_4=2.591, \qquad k_4=\lambda=0.5959$$

and

$$y_4=d_4d_2x(1+x^2)/[1+(2+d_2^2)x^2+x^4]$$
$$=15.68x(1+x^2)/[1+38.62x^2+x^4]$$

as before.

VII. Alternative Phase Shift Networks

The preceding discussion has been based on conventional constant resistance phase shift networks with

[28] In the general case of such a decomposition of a phase shift network the individual a values obtained are not necessarily real but may occur in conjugate complex pairs. Then the two corresponding elementary sections can be combined to one physical section with normalized series arm reactance $ax/(1-bx^2)$ where $a^2<4b$. However, in the case of phase-splitting networks, complex $\bar{a}$ values do not occur if a Taylor or Tchebycheff approximation is used for the performance curve, and they do not seem to occur in other good approximations. On the other hand, their occurrence is the rule in filter design.

series arm reactances X and lattice arm reactances $-R_0^2/X$, inserted between equal resistances R_0. Then the phase shift is $\beta = 2\tan^{-1} X$. It is possible to alter one of these resistances without altering the phase shift; then a basic flat loss occurs. This is indicated in Fig. 7. Marrison[29] has shown that it is possible to replace the two lattice arm reactances by resistances R_0 without altering the phase shift (see Fig. 8). Then, if the

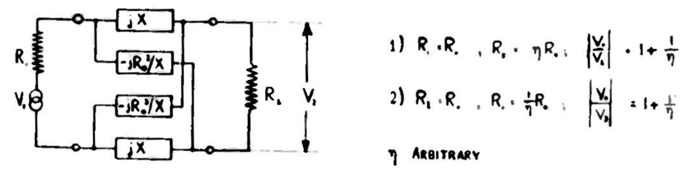


Fig. 7—Classical phase shift lattice network.

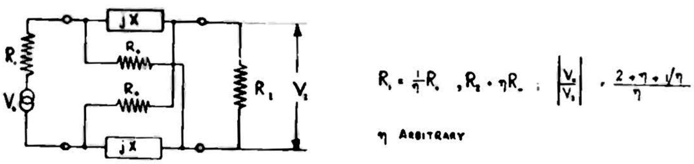


Fig. 8—Alternative to network in Fig. 7.

source and load resistance are both equal to R_0, a flat loss of 6 db is produced. It is possible to make the source resistance $(1/\eta)R_0$ and the load resistance ηR_0. Then an additional flat loss depending on η is produced, but the phase shift is still unaltered. Saraga[30] has shown that it is possible to replace one of the two remaining reactive arms by a resistance R_0 (see Fig. 9) without

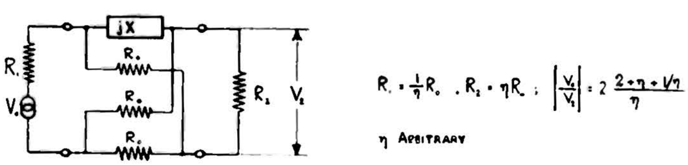


Fig. 9—Alternative to network in Fig. 7.

altering the phase shift. Then, if $\eta = 1$, the basic loss is 12 db instead of 6 db. Two other types of phase shift networks, one due to Nyquist and described by Sandeman[31] (see Fig. 10) and the other described by Wald[32] (see Fig. 11) can be shown to be special cases of the network in Fig. 9, but with more reactive elements than necessary for producing the phase shift actually produced. The circuit in Fig. 9 can be replaced by a hybrid circuit (see Sandeman). Dome[10] and Luck[12] have described a number of so-called half-lattice networks which are driven from a balanced source.

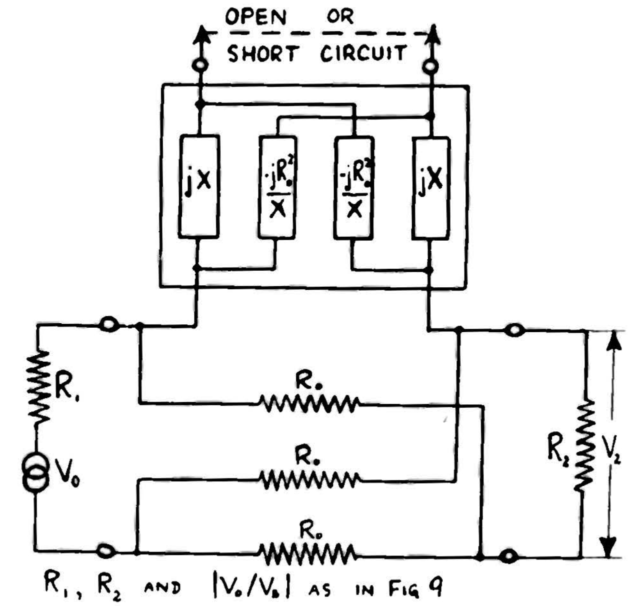


Fig. 10—Alternative to network in Fig. 7.

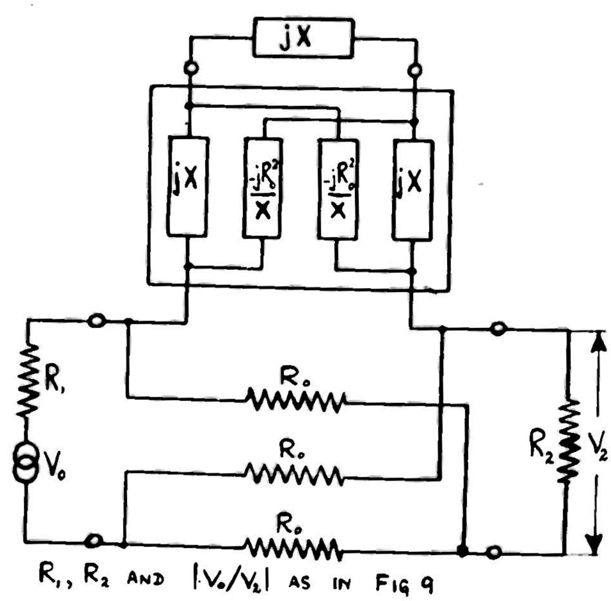


Fig. 11—Alternative to network in Fig. 7.

VIII. Dissipation-Compensated Phase Shift Networks

The effect of dissipation in the elements of a phase shift network is to distort the phase characteristic and

[29] W. A. Marrison, United States Patent No. 1,926,877, dated September 12, 1933.
[30] W. Saraga, British Patent No. 594,431, dated May 29, 1945, and U.S. Patent Application No. 670,264.
[31] E. K. Sandeman, "Phase compensation," *Elec. Commun.*, vol. 7, pp. 309–315; April, 1929.
[32] M. Wald, "Eine Kunstschaltung zur Verdreifachung des Winkelmasses eines Kreuzgliedes und ihre Anwendung zum Phasenausgleich in Pupinleitungen," *Elekt. Nach.* vol. 19, pp. 196–199; October, 1942.

to produce an attenuation varying with frequency. If the Q-values of the different components are not the same, the impedance is also affected. Starr[33] has described methods for approximate compensation of these effects of dissipation. Darlington[27] and Bode[34] have described methods for perfect compensation of the effects of dissipation. The networks are designed to meet predistorted specifications which are obtained from the original ones by assuming the occurrence of negative dissipation; then positive dissipation produces the required performance. A different method of obtaining dissipation-compensated phase shift networks will be described here.

Since any phase shift network can be built as a tandem combination of one- and two-parameter phase shift networks, it is sufficient to consider the dissipation compensation of such networks. The basic idea of the method is to consider only networks which contain a resistance in series with each inductance and a resistance in parallel with each capacitance so that these resistances can take up the dissipation resistances of reactive elements, and to design these networks so that they have the required phase characteristic β and a flat loss α_0. For the lattice network in Fig. 12 the transfer constant $\theta=\alpha+j\beta$ is given by $\tanh \frac{1}{2}\theta=Z/R_0$.

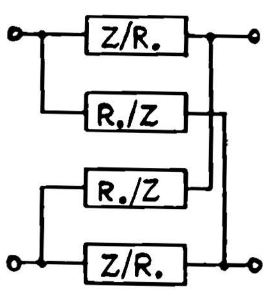


Fig. 12—Lattice network, shown for reference purposes in conjunction with Figs. 13 and 14.

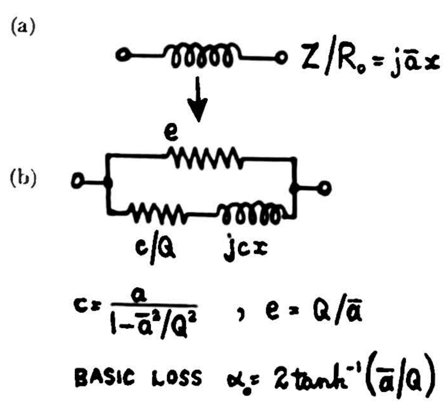


Fig. 13—Dissipation compensation of one-parameter phase shift network.

[33] A. T. Starr, British Patent No. 342,407, dated October 30, 1929.
[34] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Company, Inc., New York, N. Y., pp. 216–218; 1945.

For the one-parameter type of phase shifting network without dissipation $Z/R_0=j\bar{a}x$ (see Fig. 13(a)), and $\tan \frac{1}{2}\beta=\bar{a}x$, $\alpha=0$. Our aim is to find an impedance Z with resistances as stated above so that $Z/R_0=(C+j\bar{a}x)/(1+jC\bar{a}x)$ where $C=\tanh \frac{1}{2}\alpha_0$ and α_0 is the basic loss of the network. It can easily be shown that the network in Fig. 13(b) represents an impedance Z of this form. Its elements will be positive if Q is not too small.

We now consider the two-parameter phase shift network (without dissipation) in which Z/R_0 is as shown in Fig. 14(a). Then

$$Z/R_0=jax/(1-bx^2) \quad \text{and} \quad \alpha=0, \quad \tan \tfrac{1}{2}\beta=ax/(1-bx^2).$$

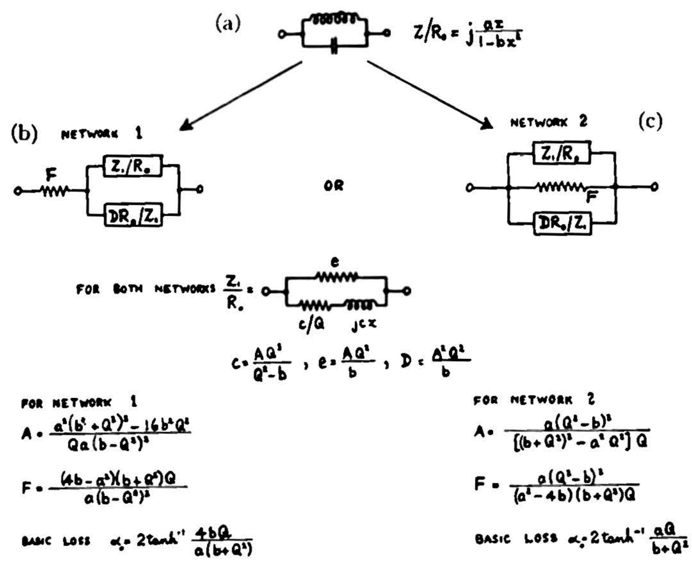


Fig. 14—Dissipation compensation of two-parameter phase shift network.

We have to find an impedance Z with resistances as stated above so that

$$Z/R_0 = [C + jax/(1 - bx^2)]/[1 + jCax/(1 - bx^2)].$$

It can be shown that the two networks shown in Figs. 14(b) and (c) have this impedance if the relations stated in these figures are satisfied. In Fig. 14(c) it is, of course, possible to absorb the resistance FR_0 in Z_1/R_0 DR_0^2/Z_1. It will be seen that the resistance FR_0 is negative in Fig. 14(b) if $a^2>4b$, and negative in Fig. 14(c) if $a^2<4b$. In Fig. 14(c) $F<0$ does not make the network necessarily nonphysical.

If $a^2\geqq 4b$ the phase shift network defined by the series arm reactance in Fig. 14(a) can be replaced by two simpler networks of the type defined by the series impedance in Fig. 13(a). As stated in Section V, footnote reference 28, only the case $a^2\geqq 4b$ seems to occur in phase-splitting problems, but the other case has been treated here too because the three transformations, described by Figs. 13 and 14, together make it possible to transform any given phase shift network into a dissipation compensated one.

Note: Since writing this manuscript the author has seen the papers by Dagnall and Rounds[35] and Farkas, Hallenbeck and Stehlick[36] in which various other methods for dissipation compensation of phase shift networks are discussed.

APPENDIX

Curve Approximation

Both equations (9e) and (10c) which give the deviation of y from unity for a given range of x and a given order n for Taylor and Tchebycheff approximations, can be written in the form

$$h(\lambda) = m + h(k), \quad \text{where} \quad m = \log_2 n,$$

and

$$h(\lambda) = \log_2 \tanh^{-1} \sqrt{\lambda}, \qquad h(k) = \log_2 \tanh^{-1} \sqrt{k}$$

in the case of (9e) and

$$h(\lambda) = \log_2 \frac{1}{[F(\lambda)]}, \qquad h(k) = \log_2 \frac{1}{[F(k)]}$$

in the case of (10c). This means that in both cases $\lambda = k_n$ as a function of k can be written in the form

$$k_n = k_2{}^m(k)$$

where $k_2{}^m(k)$ means the mth iteration of the function $k_2(k)$. Here "mth iteration" refers not only to integral values but also to fractional values of m, since $m = \log_2 n$ is only integral if n is an integral power of 2. Some discussions of the concept of non-integral iteration of functions have been given by Haldane,[37] Silberstein,[38] Hadamard.[39]

It is not possible to interpret in the same way the approximating function y as an iterated function of k because y is a function not only of k, but of x and k. However, if we generalize the concept of iteration so as to apply to functions of two variables (see Boole[40]), then y_n can be regarded as the mth iteration of $y_2(x, k)$ where $m = \log_2 n$. This will now be shown.

Since $y_2(x, k)$ leads from two independent to one dependent variable, an iteration is only possible if we introduce a second dependent variable, say, an arbitrary function $z_2(x, k)$. Then we shall define as $(y_2)^2$ and $(z_2)^2$ the functions $(y_2)^2 = y_2(y_2, z_2)$ and $(z_2)^2 = z_2(y_2, z_2)$. Furthermore, we can define iterated functions $y_2{}^m$ and $z_2{}^m$, for integral as well as non-integral values of m, as functions $y_2{}^m = F(x, k, m)$ and $z_2{}^m = G(x, k, m)$ of three variables which satisfy the following relations.

$$\left.\begin{aligned} &F(x, k, 1) = y_2(x, k), \qquad G(x, k, 1) = z_2(x, k) \\ &F[F(x, k, m_1), G(x, k, m_1), m_2] = F[x, k, (m_1+m_2)] \\ &G[F(x, k, m_1), G(x, k, m_1), m_2] = G[x, k, (m_1+m_2)] \end{aligned}\right\}. \quad (16)$$

Now if we choose as arbitrary function $z_2(x, k)$ the function $k_2(k)$—which happens to be independent of x—we see that the index law (equations (12)) can be expressed in the form of equations (16) if $m = \log_2 n$ as before. In other words:—y_n and k_n can be interpreted as the mth iteration of y_2 and k_2 when regarded as a pair of functions of x and k.

It is interesting to note that such an interpretation is also possible if, instead of an approximation by a rational function, the approximation by a polynomial is under consideration. If $y = 0$ is to be approximated by the polynomial $y_n = A_0 + A_1 x + A_2 x^2 + \cdots + x^n$ in the range $x = -\eta$ to $x = +\eta$, the nth order Tchebycheff approximation is

$$y_n = (\eta^n / 2^{n-1}) \cos [n \cos^{-1} (x/\eta)],$$

and the nth order deviation $\eta_n = 2^{1-n} \eta^n$. It is easy to show that $y_n = y_2{}^m(x, \eta)$ and $\eta_n = \eta_2{}^m(x, \eta) = \eta_2{}^m(\eta)$. These and other questions connected with non-integral functional iteration are treated in a mathematical paper by the author which is being prepared for publication.

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[35] C. H. Dagnall and P. W. Rounds, "Delay equalization of eight-kilocycle carrier programme circuits," *Bell Sys. Tech. Jour.*, vol. 28, pp. 181–195; April, 1949.

[36] F. S. Farkas, F. J. Hallenbeck, and F. E. Stehlick, "Band pass filter, band elimination filter and phase simulating network for carrier programme systems," *Bell Sys. Tech. Jour.*, vol. 28, pp. 196–220; April, 1949.

[37] J. B. S. Haldane, "On the non-linear difference equation $\Delta x_n = k\phi(x_n)$," *Proc. Cambridge Phil. Soc.*, vol. 28, part II, pp. 234–243; 1932.

[38] L. Silberstein, "Construction of groups of commutative functions," *Phil. Mag.*, pp. 43–54; January, 1945.

[39] J. Hadamard, "Two works on iteration and related questions," *Bull. Amer. Math. Soc.*, vol. 50, pp. 67–75; February, 1944.

[40] G. Boole, "A Treatise on the Calculus of Finite Differences," Macmillan and Co., London, 3rd Ed., p. 17; 1880.